Mathematics 392 – Seminar in Computational Commutative Algebra
Problem Set 9 – *Due*: Friday, November 3

From Chapter 3, section 6 in the text: 1, 2, 3, 4, 7.

Additional Problem

In this problem, you will see that there is a large connection between possible methods for computing the resultant $\text{Res}(f, g, x)$ and the division and Euclidean algorithms in $k[x]$. These connections also give an interesting alternate formula for the resultant when the roots of f are known (see part F below). (This is a "streamlined" version of problems 14 - 17 from Chapter 3, section 5. You can look at those if you need more hints, details about what to prove, etc.)

A) First, our derivation of the Sylvester matrix and the determinant formula for the resultant assumed

$$f = a_0x^\ell + a_1x^{\ell-1} + \cdots + a_\ell$$
$$g = b_0x^m + b_1x^{m-1} + \cdots + b_m$$

where neither f or g was constant. Show that if g is constant, say $g = b_0$, then the definition of the Sylvester matrix still makes sense and yields $\text{Res}(f, b_0, x) = b_0^\ell$.

B) Suppose that $\deg(f) = \ell < m = \deg(g)$. Show that if we divide g by f using the one-variable division algorithm, yielding $g = qf + r$, then

$$\text{Res}(f, g, x) = a_0^{m-\deg(r)}\text{Res}(f, r, x).$$

Hint: Look at what happens if you do one step of division and replace g by $g_1 = g - (b_0/a_0)f$. Do the corresponding column operation on all the "g-columns" in the Sylvester matrix, and expand the determinant along the first row. Then continue in the same fashion for all the subsequent steps in the division.)

C) Show that $\text{Res}(f, r, x) = (-1)^{\ell m}\text{Res}(r, f, x)$.

D) If we now alternate the division steps from part B and the "swap" steps from part C, show that we can compute resultants by performing the steps of the Euclidean algorithm and keeping track of the leading coefficients. (This method is *much* more efficient than computing the determinant of the Sylvester matrix when ℓ, m are large!)

E) Carry out the Euclidean algorithm and use the method suggested by part D to compute

$$\text{Res}(x^2 + x + 1, x^5 - 3x^4 - 2x^3 + 3x^2 + 7x + 6, x).$$

F) (*Extra Credit*) Show in general that if $f = a_0 \prod_{i=1}^{\ell}(x - \xi_i)$ has ℓ distinct roots in k, then

$$\text{Res}(f, g, x) = a_0^m \prod_{i=1}^{\ell} g(\xi_i)$$

(the product of the values of g at the roots of f – the same formula even works if f has multiple roots, but some different ideas are required to prove it in that case). Hint: Look for a connection between the algorithm from part D and the right-hand side here.