Mathematics 392 – Seminar in Computational Commutative Algebra
Comment/Solution for Problem Set 2
September 18

1.5/13. This problem studies the "formal derivative" of a polynomial

$$f = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \in k[x]. \tag{1}$$

Without using limits or any other properties of the real number system, we can define

$$f' = na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots + a_{n-1}$$

in a purely algebraic fashion. So this actually makes sense for polynomials with coefficients in *any field*. The surprise (?) is that all of the usual rules for derivatives still work.

a) Let $a \in k$. Then

$$(a \cdot f) = (aa_0)x^n + (aa_1)x^{n-1} + \cdots + (aa_{n-1})x + (aa_n)$$

So by (1)

$$\begin{aligned}(a \cdot f)' &= n(aa_0)x^{n-1} + (n-1)(aa_1)x^{n-2} + \cdots + (aa_{n-1}) \\ &= a(na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots + a_{n-1}) \\ &= a \cdot f'\end{aligned}$$

b) Let f and g be any two polynomials, say with $\deg f \geq \deg g$. By adding terms with zero coefficients at the start of g, we can write both using the same range of exponents:

$$\begin{aligned}f &= a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \\ g &= b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n \\ \Rightarrow (f+g) &= (a_0+b_0)x^n + (a_1+b_1)x^{n-1} + \cdots + (a_{n-1} + b_{n-1}x + (a_n + b_n) \\ \Rightarrow (f+g)' &= n(a_0+b_0)x^{n-1} + (n-1)(a_1+b_1)x^{n-2} + \cdots + (a_{n-1} + b_{n-1})\end{aligned}$$

Using the distributive law for multiplication over addition (valid in all fields), and regrouping the terms, this is

$$na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots + a_{n-1} + nb_0x^{n-1} + (n-1)b_1x^{n-2} + \cdots + b_{n-1} = f' + g'.$$

c) As some of you know, I'm a big beliver that 80% *or so of doing mathematics is really being lazy in creative ways(!)* To prove the product rule $(fg)' = f'g + g'f$ in a reasonable (i.e. creatively lazy(!)) fashion, let's notice that parts a and b combined say that the formal derivative mapping

$$\begin{aligned}' : k[x] &\to k[x] \\ f &\mapsto f'\end{aligned}$$

is *linear*, under the usual vector space structure on $k[x]$. If $f = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ and $g = b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m$, then the product fg is the sum of all the monomial products

$$a_ix^{n-i} \cdot b_jx^{m-j} = a_ib_jx^{n+m-i-j} \tag{2}$$

Apply the definition of the formal derivative to this one-term polynomial:

$$\begin{aligned}(a_ib_jx^{n+m-i-j})' &= (n+m-i-j)a_ib_jx^{n+m-i-j-1} \\ &= (n-i)a_ib_jx^{n-i-1} \cdot x^{m-j} + (m-j)a_ib_jx^{n-i} \cdot x^{m-j-1} \\ &= \left(a_ix^{n-i-1}\right)' \cdot \left(b_jx^{m-j}\right) + \left(a_ix^{n-i}\right) \cdot \left(b_jx^{m-j}\right)'\end{aligned} \tag{3}$$

In other words, the product rule works for the single term (2) in the product.

Now we use the linearity of the formal derivative. For the entire polynomials f, g, $(fg)'$ is the derivative of the sum of the terms (2) over all $0 \le i \le n$ and $0 \le j \le m$. By (2) and linearity of derivatives, $(fg)'$ is the sum of the terms in the last line of (3) for the same i and j. It is easy to see that we get $f'g + gf'$. This is what we wanted to show.