

General Information and Ground Rules

Today’s lab session, and the problems for the Midterm Problem Set will focus on two applications of elimination: the *implicitization* problem for parametric surfaces, and finding the *envelope* of a family of curves in the plane. These topics are discussed in sections 3 and 4 of Chapter 3 in “IVA”. You should submit solutions for the First Problems/Examples I, II as well as problems III, IV (Maple is not needed for this one), and V for the Midterm Problem Set, which is due on Friday, October 20.

This is an open book, open notes, open Swords 335 assignment. You must work *independently* on these problems, and you may not consult with other faculty members (e.g. Gareth Roberts) or students. You may consult other sources in the library, on the Internet, etc. but if you find something relevant there, please *footnote it*. When you have finished your solutions, please write out a short statement that you have abided by these ground rules, sign it, and return it with your solutions.

First Problems/Examples

I. Consider the parametric surface in \mathbf{C}^3 defined by

$$\begin{aligned}x &= uv + v \\y &= u^2 - 2v \\z &= v^2 + u - v\end{aligned}$$

A. Compute a *lex* Gröbner basis for the ideal

$$I = \langle x - uv - v, y - u^2 + 2v, z - v^2 - u + v \rangle$$

with the variables arranged to eliminate u, v .

- B. Use your answer to part A to find a variety $V \subset \mathbf{C}^3$ containing the image of the parametrization mapping.
- C. Use the Extension Theorem from class on Wednesday to prove that every point of V is in the image of the parametrization mapping over \mathbf{C} . (Hints: This is equivalent to showing that every partial solution (x, y, z) in $V(I_2) \subset \mathbf{C}^3$ extends to a full solution (x, y, z, u, v) in $V(I) \subset \mathbf{C}^5$. The Gröbner basis is complicated, but you’ll find what you need if you look hard enough!)

II. (Envelopes) A family of curves in the plane is given by an equation $F(x, y, t) = 0$. Setting t equal to a constant gives one of the curves in the family. In this problem, we will study the family defined by the equation:

$$F(x, y, t) = (x - t)^2 + (y - t^2)^2 - 1 = 0$$

From the form, we see that each curve $F(x, y, t_0)$ is a circle of radius 1 whose center is a point on the parabola (t, t^2) . We can visualize the curves in such a family in Maple by generating an *animated* plot where t is the “animation variable”. It is most convenient to do this in parametric form though, so we will consider the parametric equations:

$$\begin{aligned}x &= t + \cos(\theta) \\y &= t^2 + \sin(\theta)\end{aligned}$$

where the $(\cos(\theta), \sin(\theta))$ parametrizes the circle, and the (t, t^2) give the coordinates of the center.

A) Load the `plots` package:

`with(plots):`

Then enter the following command:

```
animate([t+cos(theta), t^2+sin(theta), theta=0..2*Pi], t=-2..2, frames=20);
```

To “play the animation”, you first left click the mouse over the graph, and select the options (forward or backward, loop or once, etc.) from the toolbar. Then right click the mouse over the graph and select play from the animate option on the plot menu.

The *envelope* of the family of curves given by $F(x, y, t) = 0$ is another curve that is *tangent* to each of the curves in the family at the intersection points of the envelope and the family curve. It is found by eliminating t between the two equations:

$$\begin{aligned}F(x, y, t) &= 0 \\ \frac{\partial}{\partial t} F(x, y, t) &= 0\end{aligned}$$

(We will discuss this in more detail in class next week.) For now, let’s take this as given and find the envelope for the family of curves above.

B) Define $F = (x - t)^2 + (y - t^2)^2 - 1$ as above, and compute an appropriate Gröbner basis to find generator(s) of the elimination ideal

$$I_1 = \langle F(x, y, t), \frac{\partial}{\partial t} F(x, y, t) \rangle \cap \mathbf{C}[x, y].$$

C) Use `implicitplot` to generate a picture of the envelope curve for this family of circles (the variety $V(I_1)$ with I_1 as in part B).

D) To understand the geometry here in more detail, it will help to generate a composite plot showing the animated family of curves, together with the implicit plot of the envelope, and also the path followed by the centers of the circles in the family. The `display` command in the `plots` package can be used to combine plots in this way. Look up the online help and see if you can figure out how it should work.

Additional Problems

III. The *Whitney umbrella* is the surface in \mathbf{R}^3 given parametrically by

$$x = uv$$

$$y = v$$

$$z = u^2$$

- A) Find the equation of the smallest variety containing the Whitney umbrella.
- B) Show that the parametrization fills up the variety over \mathbf{C} but not over \mathbf{R} . Over \mathbf{R} , exactly what points are omitted?
- C) Show that the parameters u, v giving a point on the Whitney umbrella are not always uniquely determined by x, y, z (that is, the parametrization is not one-to-one). Find the points where uniqueness fails, and explain your answer using a plot of the surface (the parametric plot will be most useful here).

IV. Prove Theorem 2 in Section 2 of Chapter 3 (the “geometric Extension Theorem”) using Lemma 1 and the Extension Theorem from Section 1.

V. You have just discovered the remnants of an ancient alien civilization in a subterranean complex. Near the entrance there are two large rooms connected by a short corridor 3 feet wide and 4 feet long. The floor plan is given in the diagram below:

In Room 1, at the point marked A, you have discovered a functional, but very fragile, robot. It has a horizontal cross-section that is a circle of diameter 2 feet. The robot will accept movement commands, but will not let itself be moved by any other means. Your job is to figure out a way (any way) to maneuver the robot from point A to point B, subject to the following constraints:

- 1) The aliens were apparently very clever mathematicians, but the robot’s control program is *very limited* – the way you must specify its motion is to give a single polynomial

parametric curve $x = f(t), y = g(t)$ that traces the path of the *center* of the robot's circular cross-section. Only one curve is allowed – no piecewise polynomial paths.

- 2) The robot will be damaged beyond repair if it runs into one of the walls. If it does, you will have destroyed the most amazing archeological discovery of all time. *So be careful!*
- 3) Fortunately, you get to “simulate” possible motions using Maple as often as you like until you find one that works, and then you can give the information to the robot.

For this problem you will need to generate a Maple worksheet showing the rooms and corridor, and a display showing that your path will keep the robot from hitting the walls at any point. And, *give a justification for why your path will not run the robot into the walls.* (By this I mean more evidence than the picture – for instance, if feasible, you could find the point on the envelope that is closest to the walls, and show that the distance at the closest point is strictly positive.)

Suggestions

- 1) A good first step will be to set up a convenient coordinate system. This will let you set up plots showing the rooms, etc. You can also experiment and determine some paths from point A to point B to try.
- 2) The discussion of envelopes from question II above is relevant here!
- 3) *Important Note:* You will need to be somewhat clever in how you apply the techniques we have discussed – if you use a “brute force” method, to compute the envelope for some “obvious” choices of the path, then you can easily end up giving Maple a Gröbner basis calculation that will use up an hour of calculation and tie up the network server without producing any results(!) There are several things you can experiment with:
 - the monomial order you use (look at the online help for MonomialOrder)
 - you can reduce the number of variables by looking at one “slice” $x = \text{const}$ or $y = \text{const}$ at a time. This generally speeds things up *a lot*, but you will need to look at several of these computations to make sure the robot follows an acceptable path.

However you approach the problem, you will need *give a justification for why your path will not run the robot into the walls.*