Mathematics 392 – Seminar in Computational Commutative Algebra
Lab Day 2/Problem Set 5 – Gröbner Bases in Maple
September 29, 2006

Background

We have now seen the definition of a Gröbner basis for an ideal, with respect to a monomial order $>$. Recall $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis for I if

- All $g_i \in I$, and
- $\langle LT_>(g_1), \ldots, LT_>(g_t)\rangle = \langle LT_>(I)\rangle$. (Equivalently, for every $f \in I$, $LT_>(f)$ is divisible by $LT_>(g_i)$ for some i.)

We will discuss an *algorithm*, called *Buchberger's algorithm* for computing a Gröbner basis from any generating set of I next week. For now, let's just learn how to use Maple's implementation of that algorithm, treating it as a sort of "black box".

Maple's `Groebner` *package*

All the procedures dealing with monomial orders, the division algorithm, Gröbner bases, etc. are contained in a package that you'll need to load first, with the command

```
with(Groebner);
```

(N.B. no umlauts in Maple, so the alternate English spelling is the name of the command!)

`NormalForm` – *Remainder on Division*

The command that performs the division algorithm is called *NormalForm*. It returns only the *remainder on division*, which can be thought of as a sort of *normal, or reduced form* for the input polynomial f with respect to the divisor polynomials $f_1, \ldots, f_s$. The format for the command is

```
NormalForm(f,PList,monorder);
```

where `f` is the polynomial you are dividing, `PList` is the list of divisors, and `monorder` specifies the monomial order. Almost any monomial order can be defined in Maple. To get the two we will use today, use

- `plex(x,y)` for the lexicographic order with $x > y$, `plex(y,x)` for lexicographic order with $y > x$, and similarly if there are more than 2 variables.
- `tdeg(x,y)` is the graded reverse lex order with $x > y$, `tdeg(y,z,x)` would give graded reverse lex with $y > z > x$, and so forth.

For example, enter

```
PList:=[x^2*y+x^2+1,x^3+x*y+2];
f:=expand((x^4*y+x^2*y^3)*PList[1]+y^2*PList[2]);
```

```
NormalForm(f,PList,plex(x,y));
```

to divide

$$f = (x^4y + x^2y^3)(x^2y + x^2 + 1) + (y^2)(x^3 + xy + 2)$$

by

$$[f_1, f_2] = [x^2y + x^2 + 1, x^3 + xy + 2],$$

using the lex order with $x > y$. *Question*: What does the result say about $\{f_1, f_2\}$ as a set of generators for the ideal $I = \langle f_1, f_2\rangle$.

`Basis` – Gröbner Basis Computation

The command for computing a Gröbner basis of an ideal generated by a given list of polynomials is called `Basis`. The format is

```
Basis(Plist,monorder);
```

where `Plist` is the list of generators for your ideal, and `monorder` is the monomial order to be used. Enter

```
Blex:=Basis(Plist,plex(x,y));
```

to compute the *lex* Gröbner basis for $I = \langle f_1, f_2\rangle$ as above (*lex* order with $x > y$).

Questions

1. What is the monomial ideal $\langle LT_{>_{lex}}(I)\rangle$ here? (Think of the definition of a Gröbner basis from last time.) How does this relate to the ideal $\langle LT_{>_{lex}}(f_1), LT_{>_{lex}}(f_2)\rangle$?
2. Find the remainder on division of the polynomial f from before with respect to `Blex`. Is this what you expect? Why?
3. Look carefully at the terms in the two polynomials in `Blex`. What do you notice? If you were trying to determine the points in the variety $V = \mathbf{V}(f_1, f_2)$, how would this Gröbner basis help? What would you do to find V?
4. To carry out the process you described in 2 in Maple, you can solve a one-variable polynomial by an approximate numerical method and find all its complex roots with a command like this:

   ```
   fsolve(Blex[1],y,complex);
   ```

   Then substitute each y-value into the other equation to find x. Do this, then find all the points in $V = V(f_1, f_2)$. Maple notes: When you call the *lex* Gröbner basis `Blex`, then `Blex[1]` is the first polynomial in the basis, and `Blex[2]` is the second. Look up the `subs` command in the on-line help to see how to substitute a value into an expression like a polynomial, if you haven't seen that before.
5. What happens if you compute a Gröbner basis for I, but with respect to a *lex order* with $y > x$? Describe the form of the polynomials you get.

6. Now compute the Gröbner basis for the same I, but with respect to the *grevlex* order with $x > y$. (Your results should be different!) Does this Gröbner basis have the same properties as the *lex* ones?
7. What is the monomial ideal $\langle LT_{>_{grevlex}}(I)\rangle$? (Try to determine the leading terms in the Gröbner basis elements "by hand" first; you can then check your work using the `LeadingTerm` command from the `Groebner` package. Note: If you call the *grevlex* Gröbner basis `Bgrev`, then `Bgrev[1]` is the first polynomial in the basis, `Bgrev[2]` is the second, and so forth.

Additional Problems

Do the following problems from section 2.8, using Maple to carry out the necessary calculations. Your worksheet(s) for these computations, together with explanations, hand calculations to set up the polynomials, etc. will be your problem set for the week.

- 2,
- 3 ("the method of Examples 2 and 3" is essentially what we did in parts 3 and 4 above; it would be good to look over the calculations in the text too!),
- 8,
- 9 (it would be nice to include graphics illustrating the varieties from 8 and 9 – see the help for `plot3d` to see how to plot parametric curves and surfaces in $\mathbf{R}^3$),
- 10