76th Annual William Lowell Putnam Mathematical Competition
December 5, 2015

- **Problem A1.** Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB.

- **Problem A2.** Let $a_0 = 1$, $a_2 = 2$ and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

- **Problem A3.** Compute
$$\log_2 \left(\prod_{a=1}^{2015} \prod_{b=1}^{2015} \left(1 + e^{2\pi i ab/2015}\right) \right)$$
Here i is the imaginary unit (that is, $i^2 = -1$).

- **Problem A4.** For each real number x, let
$$f(x) = \sum_{n \in S_x} \frac{1}{2^n},$$
where S_x is the set of positive integers n for which $\lfloor nx \rfloor$ is even. What is the largest real number L such that $f(x) \geq L$ for all $x \in [0, 1)$? (As usual $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

- **Problem A5.** Let q be an odd positive integer, and let N_q denote the number of integers a such that $0 < a < q/4$ and $\gcd(a, q) = 1$. Show that N_q is odd if and only if q is of the form p^k with k a positive integer and p a prime congruent to 5 or 7 modulo 8.

- **Problem A6.** Let n be a positive integer. Suppose that A, B, M are $n \times n$ matrices with real entries such that $AM = MB$, and such that A and B have the same characteristic polynomial. Prove that $\det(A - MX) = \det(B - XM)$ for every $n \times n$ matrix X with real entries.

- **Problem B1.** Let f be three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct zeroes. Show that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeroes.

- **Problem B2.** Given a list of the positive integers $1, 2, 3, 4, \ldots$, take the first three numbers $1, 2, 3$ and their sum 6 and cross all four numbers off the list. Repeat with the three smallest remaining numbers $4, 5, 7$ and their sum 16. Continue in this way, crossing off the three smallest remaining numbers and their sum, and continue the sequence of sums produced: $6, 16, 27, 36, \ldots$. Prove or disprove that there is some number in this sequence whose base 10 representation ends in 2015.

- **Problem B3.** Let S be the set of all 2×2 real matrices

  $$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  whose entries a, b, c, d (in that order) form an arithmetic progression. Find all matrices M in S for which there is some integer $k > 1$ such that M^k is also in S.

- **Problem B4.** Let T be the set of all triples (a, b, c) of positive integers for which there exist triangles with side lengths a, b, c. Express

  $$\sum_{(a,b,c)\in T} \frac{2^a}{3^b 5^c}$$

  as a rational number in lowest terms.

- **Problem B5.** Let P_n be the number of permutations π of $\{1, 2, \ldots, n\}$ such that

  $$|i - j| = 1 \text{ implies } |\pi(i) - \pi(j)| \leq 2$$

  for all i, j in $\{1, 2, \ldots, n\}$. Show that for $n \geq 2$, the quantity

  $$P_{n+5} - P_{n+4} - P_{n+3} + P_n$$

  does not depend on n and find its value.

- **Problem B6.** For each positive integer k, let $A(k)$ be the number of odd divisors of k in the interval $[1, \sqrt{2k})$. Evaluate

  $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{A(k)}{k}.$$