Mathematics 376 – Mathematical Statistics
Problem Set 8
due: Friday, April 20, 2012

A) Assume we have a linear statistical model

$$Y = \beta_0 + \beta_1 x + \epsilon$$

and ϵ is *normally distributed* with mean $\mu = 0$, and variance σ^2.

1) Given observations $(x_1, y_1), \ldots, (x_n, y_n)$ for values $x = x_i$ (known exactly), explain why the likelihood of these observations (as a function of β_0 and β_1) is given by

$$L(\beta_0, \beta_1) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(\left(-\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2\right)\frac{1}{2\sigma^2}\right)$$

2) Find the maximum likelihood estimators for β_0 and β_1 by computing the appropriate partial derivatives of $\ln(L)$, setting equal to zero, and solving for β_0, β_1. (You should get the same formulas as we obtained for the least squares estimators.)
3) Show that your answer in part 2 is really a maximum of L or $\ln(L)$ using the Second Derivative Test for functions of two variables.

B) In class, by solving the normal equations using Cramer's Rule, we obtained the following formulas (all summations extend from $i = 1$ to $i = n$, so we omit limits of summation for simplicity):

$$\widehat{\beta}_0 = \frac{(\Sigma y_i)\left(\Sigma x_i^2\right) - (\Sigma x_i)(\Sigma x_i y_i)}{n\left(\Sigma x_i^2\right) - (\Sigma x_i)^2} \tag{1}$$

and

$$\widehat{\beta}_1 = \frac{n(\Sigma x_i y_i) - (\Sigma x_i)(\Sigma y_i)}{n\left(\Sigma x_i^2\right) - (\Sigma x_i)^2} \tag{2}$$

Recall that in class we also introduced the quantities

$$S_{xx} = \sum(x_i - \overline{x})^2 \qquad S_{xy} = \sum(x_i - \overline{x})(y_i - \overline{y})$$

Show that (2) is equivalent to

$$\widehat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$$

and then that (1) is equivalent to

$$\widehat{\beta}_0 = \overline{y} - \widehat{\beta}_1 \overline{x}$$

From the text: Chapter 11/4 (show all work for calculations by hand, check with `R`),9 (show all work for calculations by hand, check with `R`), 14 (can be done entirely in `R`), 69 (can be done entirely in `R`).