Mathematics 375 – Probability and Statistics 1
Discussion 2 – The "Method of Moment-Generating Functions"
December 2, 2011

Background

Because of the Uniqueness Theorem (Theorem 6.1 in our text), whenever we can compute the moment-generating function for a random variable and recognize it as one of our standard forms, then we know its distribution. As a result, we know its probability density function, mean, variance, and hence "everything about it"(!) Today, we want to use this idea to work several examples and identify what we have.

Here are two useful facts:

1. From a problem set, we know that if $U = aY + b$, where a, b are constants, then

$$m_U(t) = e^{bt} m_Y(at)$$

2. If $Y_1, \ldots, Y_n$ are independent and $U = Y_1 + \cdots + Y_n$, then

$$m_U(t) = \prod_{i=1}^{n} m_{Y_i}(t) = m_{Y_1}(t) \times \cdots \times m_{Y_n}(t)$$

Discussion Questions

A) Prove "useful fact 2" above in the case that $Y_1, \ldots, Y_n$ are jointly continuous by writing out $E(e^{tU})$ in terms of $Y_1, \ldots, Y_n$. Use the fact that if $Y_1, \ldots, Y_n$ are independent, then their joint density has the form

$$f(y_1, \ldots, y_n) = f_1(y_1) \times \cdots \times f_n(y_n),$$

where $f_i(y_i)$ are the marginal densities.

B) Next, we will verify one point that we deferred in discussing the use of the standard normal table. Recall, we said that if Y is normal with mean μ and standard deviation σ, then

$$Z = \frac{Y - \mu}{\sigma}$$

would have a standard normal distribution (i.e. normal with mean 0 and standard deviation 1). We never actually justified this claim before, except in a review problem for Exam 2. Note that

$$Z = \frac{1}{\sigma} Y - \frac{\mu}{\sigma}.$$

Find the moment generating function of Z given the moment generating function for the normal Y:

$$m_Y(t) = e^{\frac{t^2\sigma^2}{2}+\mu t}.$$

and the "useful fact" 1 above. Deduce that Z must have a standard normal distribution.

C) In class we showed that if $Z \sim Normal(0,1)$ is a standard normal, then Z^2 has a χ^2 distribution with one degree of freedom, by computing its moment-generating function. Suppose $Z_1, \ldots, Z_k$ are independent random variables, each with a standard normal distribution.

1. What is the distribution of $X = Z_1 + \cdots + Z_k$?
2. What is the distribution of $U = Z_1^2 + \cdots + Z_k^2$?

Assignment

Group writeups due in class by Monday, December 5.