Mathematics 375 – Probability and Statistics 1
Solutions for Exam 1 – October 10, 2003

Directions Do all work in the blue exam booklet. There are 100 regular points and 10 extra credit points.

I. (20) A manufacturer of electronic components tests the lifetimes of a certain type of battery and finds the following data:

$$123, 116, 122, 110, 125, 126, 111, 118, 117, 120$$

(lifetimes in hours). How many of the sample points are within one standard deviation of the sample mean? Is there reason to believe the lifetime of this type of battery is not normally distributed from this small sample? Explain.

Solution: The sample mean is

$$\overline{x} = \frac{1}{10}\sum_{i=1}^{10} x_i = 118.8$$

The sample standard deviation is

$$s = \sqrt{\frac{1}{9}\sum_{i=1}^{10}(x_i - \overline{x})^2} \doteq 5.47$$

There are exactly 6 out of the 10 data points within 1 standard deviation of the mean: $123, 116, 122, 118, 117, 120$. This is quite close to the 68% we would expect for a normal distribution. Especially with such a small sample, there is no reason to suspect that the battery lifetimes are not normally distributed.

II. (15) In a regional spelling bee, the 10 finalists consist of 5 girls and 5 boys. Assume that all the finalists are equally proficient spellers and that the outcome of the contest is random. What is the probability that 4 of the top 5 finishers will be female?

Solution: This can be done either by considering the five top finishers as an ordered list or as an unordered list – there are exactly 5! orderings and that "cancels out" in the ratio for the probability. If we think of the five top finishers as a "committee" (unordered) of 4 girls and 1 boy selected from the 10 finalists, the probability is $\binom{5}{4}\binom{5}{1}/\binom{10}{5}$. (Note this is one value of a hypergeometric pdf!).

III.
A) (15) State and prove the Law of Total Probability.

Solution: See class notes.

B) (10) The Podunk City police department plans a crackdown on speeders by placing radar traps at four different locations L_1, L_2, L_3, L_4. The probability that each of the traps is manned at any one time is .4, .3, .2, .3 respectively (and the police really mean business – if a trap is manned every speeder who passes it will get a ticket). Speeders have probabilities of passing the four locations of .2, .1, .5, .2 respectively, and no one passes more than one. What is the probability that a given speeder will actually receive a ticket?

Solution: Let T be the event that the speeder gets a ticket, and let L_i be the event that the speeder passes location i. The second set of given information .2, .1, .5., 2 are the $P(L_i)$. The first are $P(T|L_i)$ since the speeder gets a ticket whenever he passes that trap and the trap is manned. Every speeder passes one and only one of the four trap locations, so they form the partition of the sample space of speeders. By the Law of Total Probability, we have

$$P(T) = \sum_{i=1}^{4} P(T|L_i)P(L_i) = (.4)(.2) + (.3)(.1) + (.2)(.5) + (.3)(.2) = .27$$

C) (10) In the situation of part B, given that a speeder received a ticket, what is the probability that he passed location L_2?

Solution: By Bayes' Rule,

$$P(L_2|T) = P(T|L_2)P(L_2)/.27 = (.3)(.1)/(.27) = .\overline{1} = 1/9.$$

IV. (10) Let A, B be events for which $P(A) = .2$, $P(B) = .3$ and $P(A \cap B) = .06$. Are $\overline{A}$ and $\overline{B}$ independent events?

Solution: By the complement rule, $P(\overline{A}) = .8$ and $P(\overline{B}) = .7$. By the sum rule $P(A \cup B) = P(A) + P(B) - P(A \cap B) = .2 + .3 - .06 = .44$. Hence $P(\overline{A \cup B}) = P(\overline{A} \cap \overline{B}) = 1 - .44 = .56$. Since $.56 = (.8)(.7)$, $\overline{A}$ and $\overline{B}$ *are independent events.*

V. An allergist knows that 30% of all people are allergic to the pollen of burdock weed.

A) (10) Starting from some point in time, the first five patients in sequence that the allergist sees are not allergic to burdock pollen. Given that, what is the probability that the first patient the doctor sees who is allergic to burdock pollen will come after the 20th patient seen? Explain the assumptions you are making to derive your solution.

Solution: Since the problem asks about the probability that the first patient the doctor sees is allergic, this situation calls for a *geometric* random variable Y. The question translates into what is $P(Y > 20|Y > 5)$. For the geometric random variable with $p = .3$, $p(y) = (.7)^{y-1}(.3)$, so using the formula for the sum of a geometric series,

$$\begin{aligned} P(Y > 20|Y > 5) &= \frac{\sum_{y=21}^{\infty}(.7)^{y-1}(.3)}{\sum_{y=6}^{\infty}(.7)^{y-1}(.3)} \\ &= \frac{(.7)^{20}}{(.7)^{5}} \\ &= .7^{15} \end{aligned}$$

B) (10) What is the probability that from 4 to 7 (inclusive) of the next 20 patients she sees are allergic to burdock pollen? Explain the assumptions you are making to derive your solution.

Solution: Here we are dealing with a binomial random variable Y with $n = 20$, $p = .3$. Using the table of binomial probabilities,

$$P(4 \le Y \le 7) = P(Y \le 7) - P(Y \le 3) = .772 - .107 = .665$$

Extra Credit (10) The Yankees won 10 of the 19 regular season games they played against the Red Sox this year. Assuming that the Yankees have a 10/19 chance of winning each game they play against the Sox, as of last Tuesday (i.e. before the first game), what was the probability that the Sox will clinch the ALCS at home? Assume the games are independent events, and recall that the series starts with two games in New York, then moves to Boston for the next 3 (if needed), then back to New York for the remainder of the series (if needed).

Answer: $\left(\frac{9}{19}\right)^4 + 4\left(\frac{9}{19}\right)^4 \frac{10}{19}$