Mathematics 375 – Probability Theory
Review Sheet, Final Exam
December 1, 2011

General Information

The final examination for this course will be given at 3:00 pm on Friday, December 16 in our regular class room, Swords 359. The exam will be roughly twice the length of one of the two midterms and I will let you continue working on it as long as you want (even after the formal end of the exam period at 5:30pm, if necessary). As was true on the midterms, I will let you bring a sheet of information for your use on the exam – for this exam, you can place anything you want on *one side of an 8.5 × 11 inch piece of paper*. I will provide copies of whatever tables from the text might be needed to work some of the problems.

Topics to be Covered

This will be a comprehensive exam – all topics we have discussed this semester are "fair game." This includes:

1) Descriptive statistics such as the mean, standard deviation, frequency histograms, etc. The "empirical rule".
2) Discrete sample spaces and counting techniques (especially in connection with the "sample point method" for the probability of an event): the $m \times n$ rule, permutations, binomial and multinomial coefficients.
3) The "event composition method" for probabilities
4) Conditional probabilities, independence of events, Bayes' Rule, the Law of Total Probability
5) Discrete random variables: probability distribution functions, cumulative distribution functions, expected values and variances of functions of a discrete random variable, moment generating functions. Know the situations leading to binomial, geometric, Poisson, and hypergeometric random variables and how to apply them.
6) Continuous random variables: probability distribution functions, cumulative distribution functions, expected values and variances of functions of a continuous random variable, moment generating functions. Know the situations leading to uniform, exponential, gamma, beta, and normal random variables and how to apply them.
7) Tchebysheff's Theorem.
8) Multivariate probability distributions: joint densities, marginal and conditional densities, expected values in this setting, conditions for independence, the covariance and the general formula for the variance of a linear combination of random variables.
9) Using moment generating functions and the uniqueness theorem to determine the distribution of a random variable.
10) The method of distribution functions to determine the distribution of a random variable.
11) Know the statement and proof of the Central Limit Theorem.

Suggestions on How to Study

Start by reading the above list of topics carefully. If there are terms there that are unfamiliar or for which you cannot give the precise definition, start by reviewing those topics. Review the class notes. *Everything on the final will be closely related to something we have discussed at some point this semester.* Also look back over your graded problem sets and exams. If there are problems that you did not get the first time around, try them again now, consulting the solutions on reserve in the Science Library as necessary. Then go through the suggested problems from the review sheets. If you have worked these out previously, it is not necessary to do them all again. But try a representative sample "from scratch" – don't just look over your old solutions. Practice thinking through the logic of how the solution is derived again.

Suggested Practice/Review Problems

Look at the problems from the two previous review sheets for the topics from Chapters 1 - 5. From Chapter 5/105 (do all the computations needed to find $V(Y_1 - Y_2)$ using the formula of Theorem 5.12), 147,149,151,161. Chapter 6/93,95,107,111.

Review Session

I will be happy to run a review session for the final exam. We can discuss a time in class on Friday, December 9.

Sample Exam Questions

Note: This was the final exam given in the section of this class taught in 2009. Solutions for these questions are available on the course homepage for that section – go to my personal homepage, follow links Course Homepages/Previous Course Home Pages, and look at the page for MATH 375 from Fall 2009.

I. Twenty students in a probability and statistics class were asked to report the number of pets in their families, giving the following data:

$$3, 3, 2, 4, 1, 0, 1, 2, 3, 5, 1, 2, 2, 1, 1, 0, 2, 4, 3, 1$$

A) (10) Construct a relative frequency histogram for this data using one "bin" for each integer value.
B) (10) How many of the data values are within two sample standard deviations of the sample mean? How does this compare with the "empirical rule?"

II. Suppose you have a key ring with N keys, exactly one of which is your car key. You are parked in a country lane on a moonless night and can't see the keys or the ring. So you try the keys in your car door lock one by one until you find the right one.

A) (10) Assuming you are not careful and you mix up all the keys after each try, what is the probability that you take at least y tries to find the right key?

B) (5) Let Y be the discrete random variable giving the number of the trial on which you find the right key. What is the expected value of Y?

C) (10) Now suppose you are more careful and you take each wrong key off the key ring and put it aside after you try it so that it does not get mixed back in with the keys you have not tried. Let Z be the discrete random variable giving the number of the trial on which you find the right key by this method. What is the expected value of Z?

III. (Hypothetical) In the city of Worcester, 50 percent of registered voters are Democrats, 30 percent are Republicans, and 20 percent are Independents. In the recent Senate primary, 15 percent of Democrats voted, 10 percent of Republicans voted, and 35 percent of Independents voted.

A) (10) If a single voter is selected at random, what is the probability that he or she voted in the Senate primary?

B) (10) Given that a registered voter did vote, what is the probability that the voter was an Independent?

IV. Let X be a random variable for which $E(X) = \mu$ and $V(X) = \sigma^2$. Let c be an arbitrary constant.

A) (10) Show that $E((X-c)^2) = (\mu - c)^2 + \sigma^2$.

B) (5) For which c is $E((X-c)^2)$ a minimum?

V. The number of customers arriving at a fast food restaurant in a typical hour has a Poisson distribution with mean 20.

A) (5) What is the probability that 17 customers arrive in a given hour?

B) (10) If the store is open 12 hours a day, what is the probability that exactly 17 customers will arrive in exactly 4 out of the 12 hours?

VI. A continuous random variable Y has p.d.f

$$f(y) = \begin{cases} c(y^2+y) & \text{if } 1 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

A) (5) What is the value of c?

B) (5) What is $P(1 \leq y \leq 3/2)$?

C) (10) Find $E(Y)$ and $V(Y)$.

VII. (15) The diameters of the bolts in a large box follow a normal distribution with mean 2 cm and a standard deviation of 0.03 cm. The diameters of the holes in the nuts in another large box are also normally distributed, but with mean 2.02 cm and standard deviation 0.04 cm. A bolt and a nut will fit together if the hole in the nut is larger than the diameter of the bolt, but the difference is no larger than .05 cm. If a nut and a bolt are selected independently and randomly from the two boxes, what is the probability that they will fit together?

VIII. Let X and Y be independent random variables with moment-generating functions

$$m_X(t) = m_Y(t) = e^{t^2+3t}$$

A) (10) What is the moment generating function of $Z = 3X + 2Y - 4$?
B) (10) What is the distribution of Z?

IX. Twenty students in a probability and statistics class take a final examination independently of one another. The number of minutes each student requires to complete the exam is a random variable with an exponential distribution with mean 120 minutes. The students all start work at 8:30am.

A) (10) What is the probability that at least one student out of the 20 will complete the exam by 10:00 am?
B) (10) What is the distribution of the total time taken by all of the students to complete the exam?

X. Suppose Y_1, Y_2 have joint density

$$f(y_1, y_2) = \begin{cases} 24y_1y_2 & \text{if } y_1 \geq 0,\ y_1 + y_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

A) (10) Are Y_1, Y_2 independent?
B) (10) What is $V(8Y_1 - 2Y_2)$?
C) (10) Use the method of distribution functions to find the density function for $U = Y_1 + Y_2$.

Extra Credit (10) One form of the Law of Large Numbers states that if $X_1, \ldots, X_n$ are independent and identically distributed random variables for which $E(X_i) = \mu$ and $V(X_i) = \sigma^2$ exist, then for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|\overline{X} - \mu| < \epsilon) = 1.$$

Prove this statement without assuming anything else about the distribution of the X_i.