Mathematics 375 – Probability Theory
Solutions – Midterm Exam 2 Practice Problems
November 9, 2011

I. If $Y$ denotes the number of underage students who get carded, then $Y$ is hypergeometric with $N = 10$, $r = 4$, $n = 5$. So

$$P(Y = 2) = \frac{\binom{4}{2}\binom{6}{3}}{\binom{10}{5}} = \frac{10}{21} \doteq .48$$

II. By examining the form of the density function, we see that $Y$ has a *beta distribution* with $\alpha = \beta = 3$. Hence:

A) The constant $c$ must be

$$c = \frac{1}{B(3,3)} = \frac{\Gamma(6)}{\Gamma(3)\Gamma(3)} = \frac{5!}{2!2!} = 30$$

B) The variance is

$$V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{3 \cdot 3}{36 \cdot 7} = \frac{1}{28}$$

C) The cumulative distribution function is the antiderivative $F(y)$ of the density with $F(-\infty) = 0$ and $F(+\infty) = 1$. This is

$$F(y) = \begin{cases} 0 & \text{if } y < 0 \\ 10y^3 - 15y^4 + 6y^5 & \text{if } 0 \le y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

D) This is

$$\int_{.1}^{.25} 30y^2(1-y)^2 \, dy = F(.25) - F(.1) \doteq .09496.$$

III. The lifetime $Y$ of a single switch has the exponential density

$$f(y) = \begin{cases} \frac{1}{2}e^{-y/2} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

A) The probability that a single switch fails during the first year is the same as the probability that its life is less than 1:

$$P(Y < 1) = \int_0^1 \frac{1}{2}e^{-y/2} \, dy = -e^{-y/2}|_0^1 = 1 - e^{-1/2} \doteq .3935$$

B) The probability that at most 30 out of 100 of these switches will fail in the first year is computed by the *binomial* probability formula (since we assume the switches operate independently):

$$\sum_{y=0}^{30} \binom{100}{y} (.3935)^y (.6065)^{100-y}.$$

IV.

A) Let $Y$ be the weight of one poodle

$$P(7.3 < Y < 9.1) = P\left(\frac{7.3-8}{.9} < \frac{Y-8}{.9} < \frac{9.1-8}{.9}\right)$$

This is the same as

$$P\left(-.78 < \frac{Y-8}{.9} < 1.22\right)$$

We know $Z = \frac{Y-8}{.9}$ is a standard normal, so we can use the *standard normal table* to find this probability. By the symmetry of the normal density,

$$P(-.78 < Z < 0) = P(0 < Z < .78) = .5 - .2177 = .2823$$

We also have

$$P(0 < Z < 1.22) = .5 - .1112 = .3888$$

Hence

$$P(-.78 < Z < 1.22) = .2823 + .3888 = .6711$$

B) Partial credit would be given for an answer like

$$E(Y^3) = \int_{-\infty}^{\infty} y^3 \frac{1}{\sqrt{2\pi(.9)^2}} e^{-(y-8)^2/2(.9)^2}\, dy.$$

But the only reasonable way to get a numerical value for this is to think of using the *moment-generating function* for $Y$. $E(Y^3) = m'''(0)$ where $m(t)$ is the mgf for a normal random variable. The general form of this is

$$m(t) = e^{\frac{\sigma^2 t}{2} + \mu t}$$

Hence differentiating with the chain and product rules:

$$\begin{aligned}
m'(t) &= e^{\frac{\sigma^2 t}{2} + \mu t}(\sigma^2 t + \mu) \\
m''(t) &= e^{\frac{\sigma^2 t}{2} + \mu t}((\sigma^2 t + \mu)^2 + \sigma^2) \\
m'''(t) &= e^{\frac{\sigma^2 t}{2} + \mu t}((\sigma^2 t + \mu)^3 + 3\sigma^2(\sigma^2 t + \mu)) \\
\Rightarrow m'''(0) &= \mu^3 + 3\sigma^2 \mu
\end{aligned}$$

With $\mu = 8$ and $\sigma = .9$, this yields $E(Y^3) = 531.44$.

V. This is the density for a Gamma-distributed random variable with $\alpha = 2$ and $\beta = 1$. Hence

$$m(t) = (1-t)^{-2}$$

(This can also be computed directly as

$$E(e^{tY}) = \int_0^\infty e^{ty} \cdot ye^{-y}\, dy.)$$

VI. From the situation, it is clear that the selection of missiles is being done *without replacement*. Hence the required probabilities will be computed using the hypergeometric formulas with $N = 10$, $r = 3$, $n = 4$. Let $Y$ be the number that *will not* fire out of the four selected

A) The probability that all four of the chosen missiles will fire is

$$P(Y=0) = \frac{\binom{3}{0}\binom{7}{4}}{\binom{10}{4}} = \frac{1}{6}$$

B) The probability that at most 2 will not fire is

$$P(Y \le 2) = \frac{\binom{3}{2}\binom{7}{2} + \binom{3}{1}\binom{7}{3} + \binom{3}{0}\binom{7}{4}}{\binom{10}{4}} = \frac{29}{30}.$$

VII.

A) The mean survival time is $E(Y) = \alpha\beta = 12$ weeks.

B) This is not possible since $12\sqrt{2} = 2\sigma$. By Tchebysheff's Theorem, $P(|Y - \mu| < 2\sigma) \ge .75$ for every random variable (independent of its distribution).

C) The probability that a single animal survives 15 or more weeks is computed like this (integrating by parts):

$$\begin{aligned} p &= \int_{15}^\infty \frac{ye^{-y/6}}{36}\, dy \\ &= \frac{-1}{6} ye^{-y/6}\Big|_{15}^\infty + \frac{1}{6}\int_{15}^\infty e^{-y/6}\, dy \\ &= \frac{5}{2}e^{-5/2} + e^{-5/2} \\ &= \frac{7}{2}e^{-5/2} \doteq .2873 \end{aligned}$$

Then if $Y$ is the number out of the 6 that survive longer than 15 weeks,

$$P(Y \ge 2) = 1 - P(Y=0) - P(Y=1) = 1 - \binom{6}{0}p^0(1-p)^6 - \binom{6}{1}p^1(1-p)^5 \doteq .5520.$$

(Note: The original posting of this solution contained an error because the Gamma density used did not incorporate the proper value of $\beta = 6$.)

VIII. Let $w$ be the warrantee period and $Y$ be the lifetime of a randomly chosen motor. We want to choose $w$ so that the probability that a motor lasts less than the warrantee period is only .03, or in other words: $P(Y < w) = .03$. Since $Y \sim Normal(10, 4)$, this is the same as

$$.03 = P(Y < w) = P(Z < \frac{w-10}{2})$$

From the standard normal table, we see $\frac{w-10}{2} \doteq -1.88$ so $w = 6.24$ years.

IX. The joint density is nonzero on the triangle in the $y_1, y_2$ plane with vertices at $(0,-1), (1,0)$, and $(0,1)$.

A) Then

$$P(Y_2 > 0) = \int_0^1 \int_0^{1-y_1} 30y_1y_2^2 \; dy_2 \; dy_1 = \frac{1}{2}$$

(This can be seen without calculation if you notice the symmetry of the region and the density function under reflection across the $y_1$-axis; otherwise just compute!)

B) The marginal density of $Y_1$ is 0 for $y_1 \notin [0,1]$ and

$$\int_{y_1-1}^{1-y_1} 30y_1y_2^2 \; dy_2 = 20y_1(1-y_1)^3$$

if $0 \le y_1 \le 1$. Since

$$B(2,5) = \frac{\Gamma(2)\Gamma(4)}{\Gamma(6)} = \frac{3!}{5} = \frac{1}{20}$$

This is the beta density with $\alpha = 2$ and $\beta = 4$.

C) The conditional density is computed as $f(y_1, y_2)/f_1(y_1)$. So by part B), this gives

$$\begin{aligned} f(Y_2|Y_1 = y_1) &= \frac{30y_1y_2^2}{20y_1(1-y_1)^3} \\ &= \frac{3y_2^2}{2(1-y_1)^3} \end{aligned}$$

for $-1 \le y_2 \le 1$ and zero otherwise. (Note: we would use this with a particular value of $y_1$ substituted in, and think of it as a function of $y_2$.)