

Mathematics 376 – Probability and Statistics II
Problem Set 8
due: Thursday, April 15, 2010

A) Assume we have a linear statistical model

$$Y = \beta_0 + \beta_1 x + \epsilon$$

and ϵ is *normally distributed* with mean $\mu = 0$, and variance σ^2 .

- 1) Given observations $(x_1, y_1), \dots, (x_n, y_n)$ of Y explain why the likelihood of these observations (as a function of β_0 and β_1) is given by

$$L(\beta_0, \beta_1) = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \exp \left(\left(- \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2 \right) \frac{1}{2\sigma^2} \right)$$

- 2) Find the maximum likelihood estimators for β_0 and β_1 by computing the appropriate partial derivatives of $\ln(L)$, setting equal to zero, and solving for β_0, β_1 . (You should get the same formulas as we obtained for the least squares estimators.)
- 3) Show that your answer in part 2 is really a maximum of L or $\ln(L)$ using the Second Derivative Test for functions of two variables.

B) In class, by solving the normal equations using Cramer's Rule, we obtained the following formulas (all summations extend from $i = 1$ to $i = n$, so we omit limits of summation for simplicity):

$$(1) \quad \hat{\beta}_0 = \frac{(\Sigma y_i)(\Sigma x_i^2) - (\Sigma x_i)(\Sigma x_i y_i)}{n(\Sigma x_i^2) - (\Sigma x_i)^2}$$

and

$$(2) \quad \hat{\beta}_1 = \frac{n(\Sigma x_i y_i) - (\Sigma x_i)(\Sigma y_i)}{n(\Sigma x_i^2) - (\Sigma x_i)^2}$$

Recall that in class we also introduced the quantities

$$S_{xx} = \sum (x_i - \bar{x})^2 \quad S_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y})$$

Show that (2) is equivalent to

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$$

and then that (1) is equivalent to

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

From the text: Chapter 11/8,12,13,69. (Note: parts of problems that call for graphing of a scatter plot and the associated regression line can be done conveniently using Maple – see handouts from course homepage).