To test the effectiveness of a seal for air leaks in automobile tires, after the seal was installed in a tire, a needle was inserted into the tire and air pressure was increased until leakage was observed. The pressures (in lb per square inch) where leakage first occurred on 10 trials were:

A) (10) Construct a relative frequency histogram for this data on the interval [86.5, 96.5], subdividing into 5 equal “bins” 

(86.5, 88.5], (88.5, 90.5], (90.5, 92.5], (92.5, 94.5], (94.5, 96.5].

Solution: The histogram should have a box of height 2/10 on the first interval, a box of height 2/10 on the second interval, a box of height 3/10 on the third interval, a box of height 1/10 on the fourth interval, and a box of height 2/10 on the last interval.

C) (15) How many of the data points are within 1 standard deviation of the sample mean?

Solution: The sample mean is

$$\mu = \frac{93 + 87 + 91 + 90 + 92 + 88 + 95 + 91 + 90 + 96}{10} = 91.3.$$

The sample variance is

$$s^2 = \frac{1}{9} \sum_{i=1}^{10} (x_i - \mu)^2 = 8.01.$$

So the sample standard deviation is

$$s = \sqrt{8.01} \approx 2.83. $$

The interval $[\mu - s, \mu + s] = [88.47, 94.13].$ Six of the 10 data values are in this interval. (Note: This is relatively close to the empirical rule of 68%.)

II. Sailors on a ship can send signal by arranging (exactly) 8 colored flags along a rope line.

A) (5) How many different signals can they send if they have flags of 8 different colors?

Solution: This is the number of permutations of 8 things, 8 at a time: 8! (factorial, not exclamation point).

B) (10) How many different signals can they send if they have 4 red, 2 green, and 2 blue flags? (There are no differences between the flags of the same color.)

Solution: The red flags can occur in any 4 of the 8 positions along the rope line. Once the reds are placed, the greens go in 2 of the remaining 4, and then the blues go in the remaining two slots. By the “$m \cdot n$ rule,” this gives

$$\binom{8}{4} \cdot \binom{4}{2} \cdot \binom{2}{2} = 420.$$
possible arrangements. Note: this is the same as the multinomial coefficient
\[
\binom{8}{4, 2, 2}.
\]

C) (10) In the situation of part B, if a random arrangement of flags is constructed, what is the probability that at least 5 other flags appear between the two blue flags?

Solution: The answer from part B will be the denominator of the ratio giving the probability we want to compute. The numerator is the number of arrangements of the 8 flags where the two blues are separated by at least 5 other flags. Note that the possible locations for the blue flags are as shown:

\[BXXXXXXBX, BXXXXXXB, \text{ or } XBXXXXXB.\]

In each case, the red and green flags can be placed in \(\binom{6}{4}\binom{2}{2}\) ways (reasoning as in part B). Therefore the probability we want is
\[
\frac{3\binom{6}{4}\binom{2}{2}}{\binom{8}{4}\binom{4}{2}} = \frac{3\binom{6}{4}}{\binom{8}{4}\binom{4}{2}}.
\]

III. Let \(A_1, A_2, A_3\) be events in a sample space \(S\). Assume that \(S = A_1 \cup A_2 \cup A_3\), where \(A_i \cap A_j = \emptyset\) if \(i \neq j\). Let \(P(A_1) = .35\), \(P(A_2) = .4\), \(P(A_3) = .25\). Finally, let \(B\) be another event with \(P(B|A_1) = .1\), \(P(B|A_2) = .2\) and \(P(B|A_3) = .05\).

A) (10) State the Law of Total Probability and use it to compute \(P(B)\).

The LTP says that if we have a partition \(S = A_1 \cup \cdots \cup A_k\) into events with \(A_i \cap A_j = \emptyset\) whenever \(i \neq j\), then for all events \(B\),
\[
P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_k)P(A_k).
\]

Applying this here:
\[
P(B) = (.1)(.35) + (.2)(.4) + (.05)(.25) = .1275.
\]

B) (5) Are \(B\) and \(A_2\) independent events? Why or why not?

Solution: No. The events \(B\) and \(A_2\) are not independent, since \(P(B \cap A_2) = P(B|A_2)P(A_2) = .08\) is not the same as \(P(B)P(A_2) = (.1275)(.4) = .051\).

C) (10) What is \(P(A_1 \cup A_2|B)\)? State the name(s) of any general rule(s) you are using.

Solution: Using the definition of conditional probabilities, \(P(A_1 \cup A_2|B) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)}\).

Using standard properties of set operations, \((A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)\).

Since \(A_1 \cap A_2 = \emptyset\), it follows that \((A_1 \cap B) \cup (A_2 \cap B) = \emptyset\) also. Therefore,
\[ P((A_1 \cup A_2) \cap B) = P(A_1 \cap B) + P(A_2 \cap B) \] by the additive rule. It follows by Bayes’ Rule that the probability we want is
\[
\frac{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)}{P(B)} = 1 - \frac{P(B|A_3)}{P(B)} = 1 - (0.098) \approx 0.92.
\]

IV.

A) (10) Let \( Y \) have a binomial distribution with parameters \( n = \) number of trials and \( p = \) success probability on each trial. Show that \( E(Y) = np \).

\textit{Solution:} See the class notes or the text.

B) (15) An army regiment has 20 different squads, 16 with 25 soldiers each, 3 with 100 soldiers each, and one with 300 soldiers, for a total of 1000 soldiers. Select a soldier at random out of the 1000 and let the random variable \( X \) equal the size of the squad to which that soldier belongs. Find the probability mass function, the expected value, and the variance of \( X \).

\textit{Solution:} The possible values with nonzero probabilities are \( x = 25, 100, 300 \). The probability that \( X \) takes one of these values is the probability that the randomly selected soldier belongs to a squad of that size:
\[
\begin{align*}
P(X = 25) &= \frac{400}{1000} = 0.4 \\
P(X = 100) &= \frac{300}{1000} = 0.3 \\
P(X = 300) &= \frac{300}{1000} = 0.3 \\
P(X = x) &= 0 \text{ for all other } x.
\end{align*}
\]

The expected value is \( E(X) = (25)(0.4) + (100)(0.3) + (300)(0.3) = 130 \). The variance is
\[
V(X) = E((X - 130)^2) = (25 - 130)^2(0.4) + (100 - 130)^2(0.3) + (300 - 130)^2(0.3) = 13350.
\]
(The standard deviation would be more meaningful here: \( \sigma = \sqrt{13350} \approx 115.5 \).

\textit{Extra Credit} (10)

Let \( X \) be a discrete random variable which takes as values all natural numbers \( k \geq 1 \). Assuming that \( E(X) \) exists, show that
\[
E(X) = \sum_{k=1}^{\infty} P(X \geq k).
\]
(Also, why do we need to add the hypothesis that $E(X)$ exists in a case like this?)

**Solution:** We start from the definition of the expected value and then regroup the terms in the sum (this is justified since the infinite series must be absolutely convergent – it has all non-negative terms!): 

\[
E(X) = \sum_{k=1}^{\infty} kP(X = k)
\]

\[
= P(X = 1) + 2P(X = 2) + 3P(X = 3) + \cdots
\]

\[
= P(X = 1) + P(X = 2) + P(X = 3) + \cdots
\]

\[
+ P(X = 2) + P(X = 3) + \cdots
\]

\[
+ P(X = 3) + \cdots
\]

\[
= P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + \cdots
\]

\[
= \sum_{k=1}^{\infty} P(X \geq k).
\]

The hypothesis on $E(X)$ is necessary for this kind of random variable since for some other $Y$, $\sum_{y=1}^{\infty} P(Y = y)$ would exist and sum to 1, but the series $\sum_{y=1}^{\infty} yP(Y = y)$ might *diverge*. An example would be a random variable with 

\[
P(Y = y) = \frac{6}{\pi^2} \frac{1}{y^2}.
\]

The series $\sum_{y=1}^{\infty} \frac{1}{y^2}$ converges (*p*-series with $p = 2$). The sum is actually $\frac{\pi^2}{6}$, and that accounts for the normalization factor in the formula for $P(Y = y)$ above. But $\sum_{y=1}^{\infty} \frac{1}{y} = \sum_{y=1}^{\infty} \frac{1}{y}$ diverges (harmonic series).