Mathematics 375 – Probability and Statistics 1
Problem Set 9, *due:* Thursday, 11/12

From Wackerly, Mendenhall, and Scheaffer: 5.11, 5.14, 5.15, 5.25, 5.26ab, 5.54, 5.59, 5.65, 5.76, 5.79, 5.80.

Additional Problem

A) As we proceed into Chapter 5, we will need to make increasingly heavy use of material on multiple integrals from MATH 241 (Multivariable Calculus). As a "warm-up" for this, and to fill in a proof for a fact we just stated in class, in this problem, we will develop a proof of the formula

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

that we used in working with Beta-distributed random variables. (Thanks to Professor Anderson for suggesting this argument to derive the formula!!)

The idea is similar to what we did in class to derive the formula

$$\int_{-\infty}^{\infty} e^{-y^2/2} dy = \sqrt{2\pi}.$$

We will take two one-variable integrals, multiply them, treat as a double integral, do coordinate transformations, and derive the desired result.

The starting point will be the product:

$$\Gamma(p)\Gamma(q) = \int_0^{\infty} s^{p-1}e^{-s}\, ds \cdot \int_0^{\infty} t^{q-1}e^{-t}\, dt \tag{1}$$

1) Show that the product on the right side in (1) is the same as the double integral

$$\int_0^{\infty}\int_0^{\infty} s^{p-1}t^{q-1}e^{-(s+t)}\, dt\, ds = \int\int_R s^{p-1}t^{q-1}e^{-(s+t)}\, dA$$

(where the region of integration R is the first quadrant in the (s,t)-plane).

2) To work with this, note that we can transform the integral by introducing a new coordinate system defined by $u = s+t; s = s$. This is a linear mapping from the (s,t) plane to the (u,s) plane defined by the matrix equation

$$\begin{pmatrix} u \\ s \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}$$

The matrix has determinant of absolute value 1, hence defines an area-preserving mapping. The area element in the new coordinate system in this case is just $dA = ds\, du$. Show that this change of coordinates leads to a new equation

$$\Gamma(p)\Gamma(q) = \int\int_S s^{p-1}(u-s)^{q-1}e^{-u}\, dA \tag{2}$$

where S is the image of R under the change of coordinate mapping.

3) Show that (2) can also be written as the following iterated integral:

$$\Gamma(p)\Gamma(q) = \int_0^\infty \int_0^u s^{p-1}(u-s)^{q-1}e^{-u}\; ds\; du \tag{3}$$

4) Now, we will do a second change of coordinates, letting $\lambda = s/u; u = u$. This coordinate change is *not* area-preserving. The area element transforms to

$$ds\; du = u\; d\lambda\; du.$$

Assuming this (we will discuss the general change of variables theorem for multiple integrals in class if you have not seen it before), show that (3) becomes

$$\Gamma(p)\Gamma(q) = \int_0^\infty \int_0^1 \lambda^{p-1}(1-\lambda)^{q-1}u^{p+q-1}e^{-u}\; d\lambda\; du \tag{4}$$

(Pay careful attention to the limits of integration on the "inside" integral. Where do they come from?)

5) Deduce the desired formula

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

from (4). Very Cool!!!!