Mathematics 375 – Probability and Statistics I
Solutions – Midterm Exam 2 Practice Problems
November 12, 2009

I. If Y denotes the number of underage students who get carded, then Y is hypergeometric with $N = 10$, $r = 4$, $n = 5$. So

$$P(Y = 2) = \frac{\binom{4}{2}\binom{6}{3}}{\binom{10}{5}} = \frac{10}{21} \doteq .48$$

II. By examining the form of the density function, we see that Y has a *beta distribution* with $\alpha = \beta = 3$. Hence:

A) The constant c must be

$$c = \frac{1}{B(3,3)} = \frac{\Gamma(6)}{\Gamma(3)\Gamma(3)} = \frac{5!}{2!2!} = 30$$

B) The variance is

$$V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{3 \cdot 3}{36 \cdot 7} = \frac{1}{28}$$

C) The cumulative distribution function is the antiderivative $F(y)$ of the density with $F(-\infty) = 0$ and $F(+\infty) = 1$. This is

$$F(y) = \begin{cases} 0 & \text{if } y < 0 \\ 10y^3 - 15y^4 + 6y^5 & \text{if } 0 \le y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

D) This is

$$\int_{.1}^{.25} 30y^2(1-y)^2 \, dy = F(.25) - F(.1) \doteq .09496.$$

III. The lifetime Y of a single switch has the exponential density

$$f(y) = \begin{cases} \frac{1}{2}e^{-y/2} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

A) The probability that a single switch fails during the first year is the same as the probability that its life is less than 1:

$$P(Y < 1) = \int_0^1 \frac{1}{2}e^{-y/2} \, dy = -e^{-y/2}|_0^1 = 1 - e^{-1/2} \doteq .3935$$

B) The probability that at most 30 out of 100 of these switches will fail in the first year is computed by the binomial probability formula (since we assume the switches operate independently):

$$\sum_{y=0}^{30} \binom{100}{y} (.3935)^y (.6065)^{100-y}.$$

IV.

A) Let Y be the weight of one poodle

$$P(7.3 < Y < 9.1) = P\left(\frac{7.3-8}{.9} < \frac{Y-8}{.9} < \frac{9.1-8}{.9}\right)$$

This is the same as

$$P\left(-.78 < \frac{Y-8}{.9} < 1.22\right)$$

We know $Z = \frac{Y-8}{.9}$ is a standard normal, so we can use the *standard normal table* (p. 792) to find this probability. By the symmetry of the normal density,

$$P(-.78 < Z < 0) = P(0 < Z < .78) = .5 - .2177 = .2823$$

We also have

$$P(0 < Z < 1.22) = .5 - .1112 = .3888$$

Hence

$$P(-.78 < Z < 1.22) = .2823 + .3888 = .6711$$

B) The only reasonable way to do this problem is to think of the moment-generating function approach. $E(Y^3) = m'''(0)$ where $m(t)$ is the mgf for a normal random variable. The general form of this is

$$m(t) = e^{\frac{\sigma^2 t}{2} + \mu t}$$

Hence differentiating with the chain and product rules:

$$\begin{aligned}
m'(t) &= e^{\frac{\sigma^2 t}{2} + \mu t}(\sigma^2 t + \mu) \\
m''(t) &= e^{\frac{\sigma^2 t}{2} + \mu t}((\sigma^2 t + \mu)^2 + \sigma^2) \\
m'''(t) &= e^{\frac{\sigma^2 t}{2} + \mu t}((\sigma^2 t + \mu)^3 + 3\sigma^2(\sigma^2 t + \mu)) \\
\Rightarrow m'''(0) &= \mu^3 + 3\sigma^2\mu
\end{aligned}$$

With $\mu = 8$ and $\sigma = .9$, this yields $E(Y^3) = 531.44$.

V. This is the density for a Gamma-distributed random variable with $\alpha = 2$ and $\beta = 1$. Hence

$$m(t) = (1-t)^{-2}$$

(This can also be computed directly as

$$E(e^{tY}) = \int_0^\infty e^{ty} \cdot ye^{-y}\ dy.)$$

VI. Let Y be the number of missiles among the 4 selected that *will fire*. Then Y has a hypergeometric distribution with $P(Y=y) = \frac{\binom{7}{y}\binom{3}{4-y}}{\binom{10}{4}}$.

A) $P(Y=4) = \frac{\binom{7}{4}\binom{3}{0}}{\binom{10}{4}}$.

B) $P(2 \le Y \le 4) = \frac{\sum_{y=2}^{4}\binom{7}{y}\binom{3}{4-y}}{\binom{10}{4}}$.

VII.

A) The mean survival time is $E(Y) = \alpha\beta = 12$.

B) First

$$E(C) = 10 + (.10)E(Y^2) = 10 + (.10)(V(Y) + (E(Y))^2) = 10 + (.10)(72+144) = 31.6.$$

Then to compute the variance of the cost:

$$\begin{aligned}
E(C^2) - (E(C))^2 &= E(100 + 2Y^2 + \frac{1}{100}Y^4) - (31.6)^2 \\
&= 100 + 2E(Y^2) + \frac{1}{100}E(Y^4) - (31.6)^2 \\
&= 100 + 2(216) + \frac{1}{100 \cdot \Gamma(2) \cdot 36}\int_0^\infty y^5 e^{-y/6}\ dy - (31.6)^2 \\
&= 532 + \frac{1}{100 \cdot 36} \cdot 6^6\Gamma(6) - (31.6)^2 \\
&= 1088.64
\end{aligned}$$

C) (15) The number N that survive is a binomial random variable, so

$$P(N \ge 2) = \sum_{n=2}^{6}\binom{6}{n}p^n q^{6-n}$$

where $p = P(Y > 15) = \frac{1}{36}\int_{15}^\infty ye^{-y/6}\ dy \doteq .287$. The probability is about .55.

VIII. Let Y = the motor lifetime, as a random variable. The warrantee length should be set at $Y = y_0$ so that the area in the *lower tail* of the normal distribution is .03: $P(Y < y_0) = .03$. Standardizing, we get $Z = \frac{Y-10}{2} < \frac{y_0-10}{2}$. From the normal table,

$\frac{y_0-10}{2} = -1.88$. So $y_0 = 10 - 3.76 = 6.24$ years. (Note that shorter warrantee periods would give even smaller probability of motor failure, but they would be less attractive to customers. So the longest warrantee period giving something less than or equal to .03 would be the one to choose to make the warrantee as attractive as possible, while satisfying the company's requirements.)

IX.

A) By a general result proved in class,

$$m_X(t) = e^{bt} m_Y(at) = e^{\lambda(e^{at}-1)+bt}$$

B) We write down the integral computing the moment generating function for $Y = Z^2$, then substitute $u = \sqrt{1-2t}z$ to integrate. The integration only yields a finite result when $1 - 2t > 0$:

$$\begin{aligned}
E(e^{tY}) &= E(e^{tZ^2}) \\
&= \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\ dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(1-2t)z^2/2}\ dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\ \frac{dz}{\sqrt{1-2t}} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\ dz \cdot \frac{1}{\sqrt{1-2t}} \\
&= 1 \cdot \frac{1}{\sqrt{1-2t}} \\
&= \frac{1}{(1-2t)^{1/2}}.
\end{aligned}$$

By the standard forms, we see that this is the moment generating function for a random variable with a $\Gamma(1/2, 2)$-distribution. This is also known as a χ^2 distribution with 1 degree of freedom.