Mathematics 375 – Probability and Statistics I
Solutions – Practice Exam 2
November 7, 2005

I. If Y denotes the number of underage students who get carded, then Y is hypergeometric with $N = 10$, $r = 4$, $n = 5$. So

$$P(Y = 2) = \frac{\binom{4}{2}\binom{6}{3}}{\binom{10}{5}}$$

II. By examining the form of the density function, we see that Y has a *beta distribution* with $\alpha = \beta = 3$. Hence:

A) The constant c must be

$$c = \frac{1}{B(3,3)} = \frac{\Gamma(6)}{\Gamma(3)\Gamma(3)} = \frac{5!}{2!2!} = 30$$

B) The variance is

$$V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{3\cdot 3}{36\cdot 7} = \frac{1}{28}$$

C) The cumulative distribution function is the antiderivative $F(y)$ of the density with $F(-\infty) = 0$ and $F(+\infty) = 1$. This is

$$F(y) = \begin{cases} 0 & \text{if } y < 0 \\ 10y^3 - 15y^4 + 6y^5 & \text{if } 0 \le y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

D) This is

$$\int_{.1}^{.25} 30y^2(1-y)^2\, dy = F(.25) - F(.1) \doteq .09496.$$

III. The lifetime Y of a single switch has the exponential density

$$f(y) = \begin{cases} \frac{1}{2}e^{-y/2} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

A) The probability that a single switch fails during the first year is the same as the probability that its life is less than 1:

$$P(Y < 1) = \int_0^1 \frac{1}{2}e^{-y/2}\, dy = -e^{-y/2}|_0^1 = 1 - e^{-1/2} \doteq .3935$$

B) The probability that at most 30 out of 100 of these switches will fail in the first year is computed by the binomial probability formula (since we assume the switches operate independently):

$$\sum_{y=0}^{30} \binom{100}{y} (.3935)^y (.6065)^{100-y}.$$

IV.

A) Let Y be the weight of one poodle

$$P(7.3 < Y < 9.1) = P\left(\frac{7.3-8}{.9} < \frac{Y-8}{.9} < \frac{9.1-8}{.9}\right)$$

This is the same as

$$P\left(-.78 < \frac{Y-8}{.9} < 1.22\right)$$

We know $Z = \frac{Y-8}{.9}$ is a standard normal, so we can use the *standard normal table* (p. 792) to find this probability. By the symmetry of the normal density,

$$P(-.78 < Z < 0) = P(0 < Z < .78) = .5 - .2177 = .2823$$

We also have

$$P(0 < Z < 1.22) = .5 - .1112 = .3888$$

Hence

$$P(-.78 < Z < 1.22) = .2823 + .3888 = .6711$$

B) The only reasonable way to do this problem is to think of the moment-generating function approach. $E(Y^3) = m'''(0)$ where $m(t)$ is the mgf for a normal random variable. The general form of this is

$$m(t) = e^{\frac{\sigma^2 t}{2} + \mu t}$$

Hence differentiating with the chain and product rules:

$$\begin{aligned}
m'(t) &= e^{\frac{\sigma^2 t}{2} + \mu t}(\sigma^2 t + \mu) \\
m''(t) &= e^{\frac{\sigma^2 t}{2} + \mu t}((\sigma^2 t + \mu)^2 + \sigma^2) \\
m'''(t) &= e^{\frac{\sigma^2 t}{2} + \mu t}((\sigma^2 t + \mu)^3 + 3\sigma^2(\sigma^2 t + \mu)) \\
\Rightarrow m'''(0) &= \mu^3 + 3\sigma^2 \mu
\end{aligned}$$

With $\mu = 8$ and $\sigma = .9$, this yields $E(Y^3) = 531.44$.

Extra Credit. This is the density for a Gamma-distributed random variable with $\alpha = 2$ and $\beta = 1$. Hence

$$m(t) = (1-t)^{-2}$$

(This can also be computed directly as

$$E(e^{tY}) = \int_0^\infty e^{ty} \cdot ye^{-y}\, dy.)$$