Mathematics 375 – Probability and Statistics 1
Problem Set 8, *due:* Friday, 11/18

From Wackerly, Mendenhall, and Scheaffer, Chapter 4: 105, 107, 108, 110, 115, 120, 122, Chapter 5: 3, 6, 9, 12.

Additional Problem

A) Starting this week, we will move into Chapter 5 of the text, which deals with joint distributions of two (or more) random variables, considered together. We will need to make increasingly heavy use of material on multiple integrals from MATH 241 (Multivariable Calculus). As a "warm-up" for this, and to fill in a proof for a fact we just stated in class, in this problem, we will develop a proof of the formula

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

that we used in working with Beta-distributed random variables. (Thanks to Professor Anderson for suggesting this argument to derive the formula!!)

The idea is similar to what we did in class to derive the formula

$$\int_{-\infty}^{\infty} e^{-y^2/2} dy = \sqrt{2\pi}.$$

We will take two one-variable integrals, multiply them, treat as a double integral, do coordinate transformations, and derive the desired result.

The starting point will be the product:

$$\Gamma(p)\Gamma(q) = \int_0^\infty s^{p-1} e^{-s}\, ds \cdot \int_0^\infty t^{q-1} e^{-t}\, dt \tag{1}$$

1) Show that the product on the right side in (1) is the same as the double integral

$$\int_0^\infty \int_0^\infty s^{p-1} t^{q-1} e^{-(s+t)}\, dt\, ds = \int\int_R s^{p-1} t^{q-1} e^{-(s+t)}\, dA$$

(where the region of integration R is the first quadrant in the (s,t)-plane).

2) To work with this, note that we can transform the integral by introducing a new coordinate system defined by $u = s+t; s = s$. This is a linear mapping from the (s,t) plane to the (u,s) plane defined by the matrix equation

$$\begin{pmatrix} u \\ s \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}$$

The matrix has determinant of absolute value 1, hence defines an area-preserving mapping. The area element in the new coordinate system in this case is just $dA = ds\ du$. Show that this change of coordinates leads to a new equation

$$\Gamma(p)\Gamma(q) = \int\int_S s^{p-1}(u-s)^{q-1}e^{-u}\ dA \tag{2}$$

where S is the image of R under the change of coordinate mapping.

3) Show that (2) can also be written as the following iterated integral:

$$\Gamma(p)\Gamma(q) = \int_0^\infty \int_0^u s^{p-1}(u-s)^{q-1}e^{-u}\ ds\ du \tag{3}$$

4) Now, we will do a second change of coordinates, letting $\lambda = s/u; u = u$. This coordinate change is *not* area-preserving. The area element transforms to

$$ds\ du = u\ d\lambda\ du.$$

Assuming this (we will discuss the general change of variables theorem for multiple integrals in class if you have not seen it before), show that (3) becomes

$$\Gamma(p)\Gamma(q) = \int_0^\infty \int_0^1 \lambda^{p-1}(1-\lambda)^{q-1}u^{p+q-1}e^{-u}\ d\lambda\ du \tag{4}$$

(Pay careful attention to the limits of integration on the "inside" integral. Where do they come from?)

5) Deduce the desired formula

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

from (4). Very Cool!!!!