Mathematics 375 – Probability and Statistics 2
Clarification of Class Notes from February 8

In our first example of using a "pivotal quantity" to derive confidence intervals this day, I wasn't very clear about exactly how you derive the formulas for the endpoints. This sheet should clarify the algebra involved.

We considered the case of sampling from a $N(\mu, \sigma^2)$ population to estimate the target parameter μ (the population mean). From our work in section 7.2 of the text, we know that with n independent samples, the quantity

$$Q = \frac{\overline{Y} - \mu}{s/\sqrt{n}},$$

(where $s = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$, the sample standard deviation) satisfies

- Q contains μ as the only unknown, and
- Q has a $t(n-1)$ distribution.

Hence we can use Q as a pivotal quantity to derive confidence intervals. Here are complete derivations of each of the three cases (being more careful about signs than I was in class):

Two-sided, $(1-\alpha) \times 100\%$ *confidence interval*

Let $t_{\alpha/2}$ be the $\alpha/2$-percentile point for the t-distribution: that is $P(T > t_{\alpha/2}) = \alpha/2$. Then by symmetry:

$$\begin{aligned}
1 - \alpha &= P\left(-t_{\alpha/2}(n-1) < \frac{\overline{Y} - \mu}{s/\sqrt{n}} < t_{\alpha/2}(n-1)\right) \\
&= P\left(-t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}} < \overline{Y} - \mu < t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\right) \\
&= P\left(-\overline{Y} - t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}} < -\mu < -\overline{Y} + t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\right) \\
&= P\left(\overline{Y} + t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}} > \mu > \overline{Y} - t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\right) \\
&= P\left(\overline{Y} - t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}} < \mu < \overline{Y} + t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\right)
\end{aligned}$$

(Note: at the next-to-last step, we multiply through by -1, which reverses all inequalities, then we switch back to write the terms increasing left to right as usual).

(turn over)

One-sided, $(1-\alpha)\times 100\%$ lower confidence bound

Note: The formula I gave for this in class *was correct*, but the way I suggested to derive it wasn't (d'oh!). We start from the statement (with t_α instead of $t_{\alpha/2}$):

$$\begin{aligned}
1-\alpha &= P\left(\frac{\overline{Y}-\mu}{s/\sqrt{n}} < t_\alpha(n-1)\right) \\
&= P\left(\overline{Y}-\mu < t_\alpha(n-1)\frac{s}{\sqrt{n}}\right) \\
&= P\left(-\mu < -\overline{Y} + t_\alpha(n-1)\frac{s}{\sqrt{n}}\right) \\
&= P\left(\mu > \overline{Y} - t_\alpha(n-1)\frac{s}{\sqrt{n}}\right)
\end{aligned}$$

One-sided, $(1-\alpha)\times 100\%$ upper confidence bound

Starting from

$$1-\alpha = P\left(-t_\alpha(n-1) < \frac{\overline{Y}-\mu}{s/\sqrt{n}}\right)$$

and proceeding as for the lower confidence bound,

$$1-\alpha = P\left(\mu < \overline{Y} + t_\alpha(n-1)\frac{s}{\sqrt{n}}\right)$$