Mathematics 375 – Probability and Statistics 1
Solutions for Review Problems, Exam 2
November 7, 2005

3.157 a) Let Y be the number of imperfections. Using the formula for Poisson probabilities, we get with $\lambda = 4$

$$P(Y \geq 1) = \sum_{y=1}^{\infty} \frac{4^y}{y!} e^{-4} = 1 - e^{-4} \doteq .982$$

For b), the idea is the same but we take $\lambda = 12$

$$P(Y \geq 1) = \sum_{y=1}^{\infty} \frac{12^y}{y!} e^{-12} = 1 - e^{-12} \doteq .999994$$

3.158/Here in an 8 square yard bolt, the expected number of imperfections is $\lambda = 32$. So $E(C) = E(10Y) = 10E(Y) = 10 \cdot 32 = 320$. $V(C) = V(10Y) = 10^2 V(Y) = 100 \cdot 32 = 3200$ so $\sigma = \sqrt{3200} = 56.57$. (We are using the formulas $E(Y) = V(Y) = \lambda$ for a Poisson random variable here.)

3.160/We have

$$m(t) = (pe^t + q)^n = \left(p\left(1 + t + \frac{t^2}{2} + \cdots\right) + q\right)^n = \left(1 + pt + \frac{pt^2}{2} + \cdots\right)^n$$

Expanding and keeping only terms with t and t^2 (all we need for the problem), we get:

$$= 1 + npt + \left(\binom{n}{2}p^2 + \frac{n}{2}p\right)t^2 + \cdots$$

Hence $\mu_1' = m'(0) = np$ and $\mu_2' = m''(0) = n(n-1)p^2 + np$. (This agrees with calculations we did in class for the binomial random variables!)

3.173 a) Hypergeometric with $N = 100, r = 40$ (the defectives), and $n = 20$, so

$$p(10) = \frac{\binom{40}{10}\binom{60}{10}}{\binom{100}{20}} \doteq .119$$

(Note: The approximate value would be obtained by expressing the binomial coefficients as factorials, cancelling as many common factors as you can, then carefully multiplying everything out – straightforward but very tedious. On an exam, of course, the binomial coefficient form would be acceptable!!) b) Using the binomial distribution (treating the selection of the sample of 20 as independent events each with a probability $p = 40/100 = .4$ of getting a defective),

$$p(10) \doteq \binom{20}{10}(.4)^{10}(.6)^{10} \doteq .117$$

Note that here the binomial probability is a good approximation to the hypergeometric probability (the exact value). The idea is that $N = 40$ is large enough so that the probabilities of getting a defective do not change that much in successive trials.

4.67/See class notes.

4.88/By inspection, this has a gamma distribution with $\alpha = 3$, $\beta = 1/2$. So $E(Y) = \alpha\beta = 3/2$ and $V(Y) = \alpha\beta^2 = 3/4$.

4.101/a) This is the form for a beta density function with $\alpha = 3$ and $\beta = 5$. Thus $c = \frac{1}{B(3,5)} = 105$ b) $E(Y) = \frac{\alpha}{\alpha+\beta} = \frac{3}{8}$.

4.110/a) Using the general formula for a uniformly distributed random variable, with $\theta_1 = 0$ and $\theta_2 = 1$, we have

$$m_Y(t) = \frac{e^t - 1}{t}$$

For the next parts we use the formula $m_{aY+b}(t) = e^{bt} m_Y(at)$ proved on Problem Set 8. b) We get:

$$m_{aY}(t) = \frac{e^{at} - 1}{at}$$

This is the mgf for a uniform r.v. on $[0, a]$. c) We get:

$$m_{-aY}(t) = \frac{e^{-at} - 1}{-at} = \frac{1 - e^{-at}}{at}$$

This is the mgf for a uniform r.v. on $[-a, 0]$. d) We get

$$m_{aY+b}(t) = e^{bt}\frac{e^{at} - 1}{at} = \frac{e^{(a+b)t} - e^{bt}}{at}$$

This is the mgf for a uniform r.v. on $[b, a+b]$.

4.128/a) The distribution function (i.e. cdf) is

$$\begin{aligned} F(y) = \int_{-\infty}^{y} f(y)\ dy &= \begin{cases} 0 & \text{if } y < -1 \\ \frac{2}{\pi}\int_{-1}^{y} \frac{dy}{y^2+1} & \text{if } -1 \le y < 1 \\ 1 & \text{if } y \ge 1 \end{cases} \\ &= \begin{cases} 0 & \text{if } y < -1 \\ \frac{2}{\pi}\arctan(y) + \frac{1}{2} & \text{if } -1 \le y < 1 \\ 1 & \text{if } y \ge 1 \end{cases} \end{aligned}$$

b)

$$E(Y) = \int_{-1}^{1} y \cdot \frac{2}{\pi}\frac{dy}{y^2+1} = 0$$

since the integrand is an odd function and the interval is symmetric about zero.

4.129/Let Y be the time to finish the exam. We want to find the z so that the standard normal $Z = \frac{Y-70}{12}$ satisfies $P(Z > z) = .1$. From the normal probability table, we see $z \doteq 1.28$, so $Y = 70 + (1.28)(12) \doteq 85.4$ minutes.

4.52/Let Y be the diameter. Then we have $Z = \frac{Y-3.0005}{.001}$ is a standard normal. The tolerance range is $3.000 \pm .002$, so we want

$$1 - P\left(\frac{2.998 - 3.0005}{.001} < Z < \frac{3.002 - 3.0005}{.001}\right) = 1 - P(-2.5 < Z < 1.5)$$

Using the symmetry of the normal density, this is the same as $1 - .927 = .073$.

4.131/Using the binomial probability formula, the answer is $\sum_{y=1}^{5} \binom{5}{y}(.927)^y(.073)^{5-y} = 1 - (.927)^5 \doteq .3155$.

4.132/"Standardize" and use the normal probability table. a)

$$P(Y < 60) = P\left(\frac{Y - 75}{12} < \frac{60 - 75}{12}\right) = .1056$$

b) $P(Y \geq 60) = 1 - .1056 = .8944$. c) $P(Y \geq 90) = P\left(\frac{Y-75}{12} \geq \frac{90-75}{12}\right) = .1056$.

4.133/By inspection, this is the density for a gamma-distributed r.v. with $\alpha = 2$ and $\beta = 1/2$. Hence a) $c = \frac{1}{\Gamma(2)(1/2)^2} = 4$. b) $\mu = \alpha\beta = 1$ and $\sigma^2 = \alpha\beta^2 = 1/2$. c) $m(t) = (1 - t/2)^{-2}$.

4.135/For a beta-distributed r.v.

$$\begin{aligned} E(Y^k) &= \int_0^1 y^k \cdot \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}\, dy \\ &= \int_0^1 \frac{y^{\alpha+k-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}\, dy \\ &= \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)} \\ &= \frac{\Gamma(\alpha+k)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+k)} \end{aligned}$$

4.136/The probability that the time T to the first arrival is greater than t_0 $P(T > t_0) = P(N = 0)$ at $t = t_0$ which is the Poisson probability

$$\frac{(\lambda t_0)^0}{0!} e^{-\lambda t_0} = e^{-\lambda t_0}$$

Hence the cdf for T is

$$F(t) = P(T \leq t) = 1 - P(T > t) = 1 - e^{-\lambda t}$$

The pdf $f(t) = F'(t) = \lambda e^{-\lambda t}$, which is exponential with $\beta = \frac{1}{\lambda}$.

4.137/With $\lambda = 10$, $\int_{.25}^{\infty} 10e^{-10t}\, dt \doteq .082$

4.142/T is exponential witth $\beta = 1/2$, so

$$\int_{.25}^{\infty} 2e^{-2y}\, dy = e^{-.5} \doteq .607.$$

4.150/We must split the integral for the mgf at 0 because the formula for $|y|$ changes there. Also in order to get convergence of these integrals, we must have $|t| < 1$, so $1 + t > 0$ and $1 - t > 0$:

$$\begin{aligned} m(t) &= \int_{-\infty}^{\infty} e^{ty} f(y)\, dy \\ &= \frac{1}{2}\int_{-\infty}^{0} e^{ty}e^{y}\, dy + \frac{1}{2}\int_{0}^{\infty} e^{ty}e^{-y}\, dy \\ &= \frac{1}{2} \cdot \frac{1}{1+t} + \frac{1}{2} \cdot \frac{1}{1-t} \\ &= \frac{1}{1-t^2} \end{aligned}$$

Hence $E(Y) = m'(0) = \frac{2t}{1-t^2}|_{t=0} = 0$.

4.151/Just use our general formulas for all this. For part a, you must show that $f(y) \geq 0$ for all y and $\int_{-\infty}^{\infty} f(y)\, dy = 1$. This follows clearly since f_1 and f_2 are densities, so the total integral of f is $a \cdot 1 + (1-a) \cdot 1 = 1$. The rest follows from the usual rules

$$E(Y) = \int_{-\infty}^{\infty} y f(y)\, dy$$

and

$$V(Y) = E(Y^2) - (E(Y))^2$$