Mathematics 375 – Probability and Statistics 1
Comments and Answers forx Discussion 1
September 16, 2005

The easiest way to "go wrong" in counting is to take an approach that ends up counting the same thing more than once and not realize you are doing that. For example, on question A part 3 of this discussion, several groups seemed to be thinking like this in counting the number of "one-pair" poker hands (possibly with some "adjustments"):

The first card in the hand can be any one of the 52 cards in the deck, then to make the pair, we take one of the other three cards of the same denomination as the first one, then three other cards of different denominations:

$$52 \cdot 3 \cdot 48 \cdot 44 \cdot 40.$$

Unfortunately, this is *not correct* for the following reasons:

1) You are getting each possible pair in $2 = 2!$ different ways in the $52 \cdot 3$. (Say the pair is the 3 of hearts and the 3 of spades. You're counting the pair once if the first card is the 3 of hearts, and also a second time if the first is the 3 of spades).
2) Similarly, you're getting every combination of three other cards in the hand in $3! = 6$ different ways depending on the order in which they are selected.

The correct answer is easier to see if you think of first picking the denomination for the pair, then the two suits of the pair, then the denominations and suits of the other three cards, taking into account that the hand is an unordered set:

$$\binom{13}{1} \cdot \binom{4}{2} \cdot \frac{1}{3!} \left(\binom{12}{1} \cdot 4 \binom{12}{1} \cdot 4 \cdot \binom{11}{1} \cdot 4 \cdot \binom{10}{1} \cdot 4 \right)$$

The probability of getting a 5-card hand with exactly one pair should be

$$\frac{\binom{13}{1} \cdot \binom{4}{2} \cdot \frac{1}{3!} \left(\binom{12}{1} \cdot 4 \binom{12}{1} \cdot 4 \cdot \binom{11}{1} \cdot 4 \cdot \binom{10}{1} \cdot 4 \right)}{\binom{52}{5}} \doteq .423$$

(You have a 42.3% chance of getting one if 5 cards are drawn randomly.)

Other Answers

A.

1) $4 \cdot 13 = 52$
2) $\binom{52}{5}$ for the unordered hands, $P_5^{52} = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ for the ordered hands. Note that
$$\binom{52}{5} = \frac{1}{5!} \cdot 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48.$$

4) $\binom{13}{5} \cdot 4$ (4 choices of suit, and $\binom{13}{5}$ different combinations of 5 denominations in that suit).

B.

1) There are $140+\binom{140}{2} = 9870$ different majors and pairs of two majors. (Note: Majoring in English and Sociology isn't different from Sociology and English.)
2) There are $26^2 + 26^3 = 18252$ different codes of two or three letters.
3) The current system can continue to be used since there are more than enough codes for all majors and pairs of majors.

C.

1) With 3 equal prizes, the probability of 2 or 3 team members winning a prize is

$$\frac{\binom{25}{2}975 + \binom{25}{3}}{\binom{1000}{3}} \doteq .0018$$

2) With 3 different prizes we should keep track who wins what:

100	35	15
team	*team*	*nonteam*
team	*nonteam*	*team*
nonteam	*team*	*team*
team	*team*	*team*

In each of the first three cases, we get $25 \cdot 24 \cdot 975$ possible sets of winners, in the last $25 \cdot 24 \cdot 23$. The probability is

$$\frac{3 \cdot 25 \cdot 24 \cdot 975 + 25 \cdot 24 \cdot 23}{1000 \cdot 999 \cdot 998}$$

If you look at this carefully, you'll see that it equals the answer to part 1. It actually *doesn't matter* for the probability, since the denominators of 6 cancel. Another way to say this: For each way of assigning winners as in part 1, there are exactly 6 ways to assign the 3 different prizes as in part 2. So the probabilities are the same either way(!)