A) Assume we have a linear statistical model

\[ Y = \beta_0 + \beta_1 x + \epsilon \]

and \( \epsilon \) is *normally distributed* with mean \( \mu = 0 \), and variance \( \sigma^2 \).

1) Given observations \((x_1, y_1), \ldots, (x_n, y_n)\) of \( Y \) explain why the likelihood of these observations (as a function of \( \beta_0 \) and \( \beta_1 \)) is given by

\[ L(\beta_0, \beta_1) = \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \exp \left( \left( -\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \right) \frac{1}{2\sigma^2} \right) \]

2) Find the maximum likelihood estimators for \( \beta_0 \) and \( \beta_1 \) by computing the appropriate partial derivatives of \( \ln(L) \), setting equal to zero, and solving for \( \beta_0, \beta_1 \). (You should get the same formulas as we obtained for the least squares estimators.)

3) Show that your answer in part 2 is really a maximum of \( L \) or \( \ln(L) \) using the Second Derivative Test for functions of two variables.

B) In class, by solving the normal equations using Cramer’s Rule, we obtained the following formulas (all summations extend from \( i = 1 \) to \( i = n \), so we omit limits of summation for simplicity):

\begin{align*}
\hat{\beta}_0 &= \frac{(\Sigma y_i) (\Sigma x_i^2) - (\Sigma x_i) (\Sigma x_i y_i)}{n (\Sigma x_i^2) - (\Sigma x_i)^2} \\
\hat{\beta}_1 &= \frac{n(\Sigma x_i y_i) - (\Sigma x_i)(\Sigma y_i)}{n(\Sigma x_i^2) - (\Sigma x_i)^2}
\end{align*}

Recall that in class we also introduced the quantities

\[ S_{xx} = \sum (x_i - \bar{x})^2 \quad S_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y}) \]

Show that (2) is equivalent to

\[ \hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} \]

and then that (1) is equivalent to

\[ \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \]

*From the text:* Chapter 11/2, 5, 6, 7, 11, 15 (Note: parts of problems that call for graphing can be done conveniently using Maple – see handout from class on 4/19).