Mathematics 242 – Principles of Analysis
Problem Set 8 – **Due:** Friday, April 11

'A' Section

1. For each of the following functions and intervals: First state whether the hypotheses of the MVT are satisfied for f. Second: If they are satisfied, determine all $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. If they are not satisfied, determine whether the conclusion of the MVT holds even so.

(a) $f(x) = \frac{4x+1}{x-1}$ on $[a, b] = [0, 2]$.
(b) $f(x) = x^{1/3}$ on $[a, b] = [0, 1]$.
(c) $f(x) = \sin(\pi x)$ on $[0, 3]$.
(d) $f(x) = \begin{cases} 1/x & \text{if } x > 0 \\ 2 & \text{if } x = 0 \end{cases}$ on $[a, b] = [0, 2]$.

2. Let $f(x) = x^3 - \lambda x^2 - \lambda^2 x$, where $\lambda > 0$ is a constant.

(a) Show that f is a 1-1 function on the interval $I = [-\lambda/3, \lambda]$.
(b) Determine $f(I)$.
(c) Let g be the inverse function of f restricted to I. What is $g'(0)$?

3. Let $f(x) = x^2 + x + 3$ on $[0, 3]$.

(a) Compute $L(f, \mathcal{P})$ and $U(f, \mathcal{P})$ for the partition $\mathcal{P} = \{0, 1, 2, 3\}$.
(b) Compute $L(f, \mathcal{P})$ and $U(f, \mathcal{P})$ for the partition $\mathcal{P}' = \{0, 1/2, 1, 2, 5/2, 3\}$ and verify that the statement of Lemma 5.1.9 holds here.

'B' Section

1. Recall that on a previous problem set, we showed that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{1}$$

(a) Show that

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0. \tag{2}$$

You may only use the fact in equation (1) above and other general facts about limits. (In other words, no L'Hopital's Rule, which we have not discussed (yet) in this course – see part (e) of question 1 in the 'B' section below.)

(b) Use the limits in (1) and (2) above to show that if $f(x) = \sin(x)$ then f is differentiable at all real c, and $f'(c) = \cos(c)$. (Hint: The best way to do this is to set up the limit for the derivative like this:

$$f'(c) = \lim_{x \to c} \frac{\sin(x) - \sin(c)}{x - c} = \lim_{h \to 0} \frac{\sin(c + h) - \sin(c)}{h}.)$$

(c) Similarly, show that if $g(x) = \cos(x)$, then g is differentiable at all real c and $g'(c) = -\sin(c)$.

2. (Applications and Extensions of the "Most Valuable Theorem")

(a) Prove that if f is differentiable on (a, b) and there exists a positive constant M such that $|f'(x)| \leq M$ for all $x \in (a, b)$, then f is uniformly continuous on (a, b).

(b) Let f be a function such that the first n derivatives $f', f'', \ldots, f^{(n)}$ exist for all $x \in \mathbf{R}$, and assume that

$$f(x_1) = f(x_2) = \cdots = f(x_{n+1})$$

for distinct $x_1, x_2, \ldots, x_{n+1} \in \mathbf{R}$. Show that $f^{(n)}(c) = 0$ for some $c \in \mathbf{R}$. (An induction proof is a possibility here. What is the base case?)

(c) Let f be differentiable and assume that f' is strictly increasing on $\mathbf{R}$. If $f(a) = f(b)$ where $a < b$, then show that $f(x) < f(a) = f(b)$ for all $a < x < b$.

(d) Let f, g be continuous on $[a, b]$ and differentiable on (a, b). Show there exists $c \in (a, b)$ such that $g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a))$. (Hint: Consider the linear combination $h(x) = g(x)(f(b) - f(a)) - f(x)(g(b) - g(a))$.

(e) Use part (d) to deduce that if f, g are differentiable on a deleted neighborhood of c with $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$, if $g'(x) \neq 0$ at all points of that deleted neighborhood and if

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L \tag{3}$$

as well. That is, show that the limit on the left side of the equation in (3) exists and equals L. (This is a basic form of L'Hopital's Rule for limits.)

3. Show that $\int_0^3 f$ exists and determine its value if

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 3 & \text{if } 1 < x \leq 3. \end{cases}$$

(That is, show that f is integrable according to our definition and determine the value of $\int_0^3 f$.)