Mathematics 242 – Principles of Analysis
Problem Set 7 – **due:** Friday, April 4

'A' Section

1. Let $f(x) = \frac{15x}{x^4+3x^2+1}$. Use the Intermediate Value and/or Extreme Value Theorems to show the following:
A) For all $k \in [-3, 3]$, there exist $c \in [-1, 1]$ such that $f(c) = k$.
B) For all k with $0 < k < 3$, there exist some $c \in (1, \infty)$ such that $f(c) = k$.
C) There is a $c \in (0, 1)$ where $f(c) = 3$
D) There is a $d \in (0, 1)$ where $f'(d) = 0$.

2. Show that there is a solution of the equation $\tan(x) = x$ in the interval $\left(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2}\right)$ for every $k \in \mathbf{Z}$.

3. Using the definition of the derivative, find the value of $f'(c)$, or say why f is not differentiable at $x = c$:

A) $f(x) = x^3 + 2x - 4$ at $c = 1$.
B) $f(x) = \sin(|x|)$ at $c = 0$. Hint: Look back at Problem Set 6, B 2.
C) The function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x > 1 \\ 2x - 1 & \text{if } x \leq 1 \end{cases}$$

at $c = 1$.
D) The function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{Q}^c \end{cases}$$

at $c = 0$.

'B' Section

1. Let f be continuous on $[0, 1]$ with $f(0) < 0$ and $f(1) > 1$. Suppose that g is another continuous function on $[0, 1]$ such that $g(0) \geq 0$ and $g(1) \leq 1$. Show that there exists some $c \in (0, 1)$ such that $f(x) = g(x)$.

2. Let f be continuous on $[a, b]$ with $f(a) < k < f(b)$. Here is a variation on our proof of the Intermediate Value Theorem.
A) Let

$$T = \{x \in [a, b] \mid f(x) > k\}.$$

Show that T has a greatest lower bound and that $f(\text{glb}(T)) = k$.
B) Will this $\text{glb}(T)$ always be the same as the c we found in our proof of the IVT with $f(c) = k$? If so, prove they are the same; if not, give a counterexample.

3. This property deals with another property of real-valued functions of a real variable sometimes called *Lipschitz continuity*.

A) Let f be a function on an interval I with the property that there exists a strictly positive constant k such that $|f(x) - f(x')| \leq k|x - x'|$ for all $x, x' \in I$ (this is the definition of Lipschitz continuity). Show that f is uniformly continuous on I.

B) The converse of the statement in part A is not true: Show that $f(x) = x^{1/3}$ is uniformly continuous on $[-1, 1]$, but there is no constant k such that $|f(x) - f(x')| \leq k|x - x'|$ for all $x, x' \in [-1, 1]$. Hint: Think slopes of secant lines to the graph $y = x^{1/3}$.