Mathematics 242 – Principles of Analysis
Problem Set 6 – **due:** Monday, March 24

'A' Section

1. Determine whether each of the following limits exists using the "big theorem" for function limits and other results from section 3.2 of the text as needed.

(a) $\lim_{x\to 1} x^2 - 5x + 3$
(b) $\lim_{x\to \frac{1}{3}} x + \frac{1}{x^2}$
(c) $\lim_{x\to 1} \frac{x^3-1}{x^2-1}$
(d) Let

$$f(x) = \begin{cases} x^{1/3}\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$$

and consider $\lim_{x\to 0} f(x)$.

2. Which of the functions in question 1 are *continuous* at the indicated c in the limits there? Explain.

3. True-False. For the true statements, give a short proof. For the false statements give a counterexample.

(a) If $\lim_{x\to 1} f(x) = e - \frac{28}{10}$, then there exists a $\delta > 0$ such that $f(x) < 0$ for all x with $0 < |x - 1| < \delta$.
(b) If $|f(x)| \leq x^3$ for all x and $\lim_{x\to 2} f(x)$ exists, then $\lim_{x\to 2} f(x) \leq 8$.
(c) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by this rule:

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ -2x & \text{if } x \text{ is irrational.} \end{cases}$$

Then $\lim_{x\to 0} f(x)$ exists and equals 0.
(d) If $f(x) < g(x)$ on a deleted neighborhood of c, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = M$, then $L < M$.

'B' Section

1. Show that your answers for parts a and d of 1 on the A section are correct using the ε, δ definition (*not* the "big theorem" or other results from Chapter 3, section 1 of the text.)

2. Assume that $\lim_{x\to c} f(x) = L$.

(a) Show that there exists a constant B and $\delta > 0$ such that $|f(x)| \leq B$ for all x in the deleted neighborhood $\{x \in \mathbf{R} \mid 0 < |x - c| < \delta\}$.
(b) Using part (a), *not* the limit product rule, show that $\lim_{x\to c}(f(x))^n = L^n$ for all integers $n \geq 1$.

(c) Assume that $f(x) \geq 0$ on some deleted neighborhood of $x = c$. Show that

$$\lim_{x \to c} \sqrt{f(x)} = \sqrt{L}.$$

(*Hint:* It may help to treat the cases $L = 0$ and $L \neq 0$ separately.)

3. In this problem you will show that

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

For $0 < \theta < \frac{\pi}{2}$, the point $P = (\cos(\theta), \sin(\theta)) = (x, y)$ lies on the arc of the unit circle $x^2 + y^2 = 1$ in the first quadrant.

(a) Let $O = (0, 0)$, $Q = (\cos(\theta), 0)$, and $R = (1, 0)$. (Draw a picture!) By considering the areas of the triangle ΔOQP and the circular sector ORP, deduce that if $0 < \theta < \frac{\pi}{2}$, then $\sin(\theta)\cos(\theta) \leq \theta$. (You may use "intuitively reasonable" facts about areas such as the statement that if one plane region $\mathcal{R}$ is completely completely contained in a second region $\mathcal{S}$, then area$(\mathcal{R}) \leq$ area$(\mathcal{S})$.)

(b) Now take the tangent line to the circle at R (a vertical line), and let $S = (1, \tan(\theta))$ be the intersection of that line and the radius OP (extended). Considering the areas of the triangle ΔORS and the sector ORP as above, explain why $\theta \leq \tan(\theta)$.

(c) Combine parts (a) and (b) to deduce that if $0 < \theta < \frac{\pi}{2}$, then

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq \frac{1}{\cos(\theta)}.$$

(d) Using the one-sided form of Theorem 3.2.9 (The Limit Squeeze Theorem), show that

$$\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1.$$

(You will need to use the fact that $\cos(\theta)$ is continuous at $\theta = 0$.)

(e) Now, for $-\frac{\pi}{2} < \theta < 0$, show that $\frac{\sin(\theta)}{\theta} = \frac{\sin(|\theta|)}{|\theta|}$ and use this to see that

$$\lim_{\theta \to 0^-} \frac{\sin(\theta)}{\theta} = 1$$

as well.

(f) Finally, explain how parts (d) and (e) combine to show the statement at the start of the problem.