Mathematics 242 – Principles of Analysis
Solutions for Problem Set 5
Due: March 14, 2014

'A' Section

1. For each of the following sequences, determine three different subsequences, each converging to a different limit. For each one, express your three subsequences as x_{n_k} for a suitably chosen (strictly increasing) index sequence n_k, and give an explicit formula for n_k as a function of k:

(a) $x_n = \sin\left(\frac{n\pi}{2}\right)$

Solution: If $n_k = 2k$, then $x_{n_k} = 0$ for all k, so that subsequence converges to 0. If $n_k = 4k+1$, then $x_{n_k} = 1$ for all k, so that subsequence coverges to 1. Finally, if $n_k = 4k+3$, then $x_{n_k} = -1$ for all k so that subsequence converges to -1. There are infinitely many other correct examples too.

(b) $x_n = \frac{n}{5} - \left[\frac{n}{5}\right]$ (as usual, $[\ ]$ denotes the greatest integer function)

Solution: Let $n_k = 5k$, then $x_{5k} = k - k = 0$ for all k, so that subsequence converges to 0. Let $n_k = 5k+1$, then $x_{5k+1} = k + 1/5 - k = 1/5$ for all k, so that subsequence converges to $1/5$. Finally, if $n_k = 5k+2$, then the subsequence x_{5k+2} converges to $2/5$. (There are similar subsequences converging to $3/5$ and $4/5$ as well.)

2. Let $x_n = \sqrt{n}$. For each of the following sequences, either express that sequence as a subsequence of the sequence x_{n_k} for some explicit (strictly increasing) index sequence n_k, or say why that is impossible:

(a) $\{2, 3, 4, 5, \ldots\}$

Solution: This is the subsequence x_{n_k} for $n_k = (k+1)^2$, $k \geq 1$.

(b) $\{\sqrt{3}, \sqrt{6}, \sqrt{9}, \sqrt{12}, \ldots\}$

Solution: This is the subsequence x_{n_k} for $n_k = 3k$, $k \geq 1$.

(c) $\{1, 2, 4, 8, 16, 32, \ldots\}$

Solution: This is the subsequence x_{n_k} for $n_k = 2^{2(k-1)}$, $k \geq 1$.

3. Let $x_n = \sin\left(\frac{n\pi}{4}\right)$ and $y_n = \cos\left(\frac{n\pi}{4}\right)$.

(a) Find a (strictly increasing) index sequence n_k such that both x_{n_k} and y_{n_k} converge.

Solution: The sequence $n_k = 8k+1$ is such an index sequence since

$$\sin\left(\frac{(8k+1)\pi}{4}\right) = \sin\left(2k\pi + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and

$$\cos\left(\frac{(8k+1)\pi}{4}\right) = \cos\left(2k\pi + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

for all $k \geq 1$.

(b) Find a second (strictly increasing) index sequence n_k such that both x_{n_k} and y_{n_k} diverge.

Solution: The sequence $n_k = k$ is such a sequence.

(c) Find a third (strictly increasing) index sequence n_k such that one of x_{n_k} and y_{n_k} converges and the other diverges.

Solution: The sequence $n_k = 4k$ is such an example since $\sin(k\pi) = 0$ for all k, but $\cos(k\pi) = (-1)^k$ does not converge.

'B' Section

1. (True or False) – If the statement is true give a proof; if it is false give a counterexample.

(a) If x_n is a sequence of strictly positive numbers converging to 0, then x_n has a strictly decreasing subsequence x_{n_k}.

Solution: This is TRUE. Here is one way to see it, constructing a strictly decreasing subsequence inductively. We start with $n_1 = 1$, so $x_{n_1} = x_1$. This is the base case. Now assume we have found $x_{n_1} > x_{n_2} > \ldots > x_{n_k}$ for $1 = n_1 < n_2 < \cdots < n_k$. Since $x_n \to 0$, given $\varepsilon = x_{n_k}$, there exists N_0 in $\mathbf{N}$ such that $x_n = |x_n - 0| < x_{n_k}$ for all $n \geq N_0$. Take $n_{k+1} = \max(N_0, n_k + 1)$. Then we have $x_{n_{k+1}} < x_{n_k}$, and $n_{k+1} > n_k$. This shows we can continue to construct a strictly decreasing subsequence.

(b) If $x_n \to 0$, then x_n contains a strictly increasing subsequence or a strictly decreasing subsequence (or both).

Solution: This is FALSE. Counterexamples are the constant sequence $x_n = 0$ for all $n \geq 1$, or any "eventually constant" sequence with $x_n = 0$ for all $n \geq n_0$ for some $n_0 \in \mathbf{N}$. (Comment: The statement would be true if we assumed $x_n \neq 0$ for all n (or even all $n \geq n_0$ for some $n_0 \in \mathbf{N}$). Any such sequence contains either infinitely many positive terms or infinitely many negative terms, or both. If there are infinitely many positive terms, we get a strictly decreasing subsequence of the positive terms by part (a) of this question. If there are infinitely many strictly negative terms, then there is a strictly increasing subsequence, as can be seen by taking negatives, using part (a), then flipping signs again.)

(c) If x_n is a monotone increasing sequence with a bounded subsequence x_{n_k}, then x_n converges.

Solution: This is TRUE. Since the whole sequence is increasing, so is the subsequence x_{n_k}. But that subsequence is bounded above, say by $a \in \mathbf{R}$. We claim that the whole sequence x_n is also bounded above by a. To see that, let n be any natural number.

Since $\{n_k\}$ is a strictly increasing sequence of natural numbers, it follows that it is not bounded above. So there is some k such that $n_k \geq n$. But then since x_n is increasing and a is an upper bound for the subsequence, $a \geq x_{n_k} \geq x_n$. It follows that $a \geq x_n$ for all n, and hence the whole sequence is bounded above. Then $\{x_n\}$ converges as well by the Monotone Convergence Theorem. (You can also show that the limit of the whole sequence must be the same as the limit of the subsequence, but that was not required.)

2. Consider the sequence $x_n = \cos(n)$ (where we think of n as an angle expressed in radians).

(a) Prove that x_n has a convergent subsequence.

Solution: Since $|\cos(n)| \leq 1$ for all $n \geq 1$, this is a bounded sequence. The statement to be proved is a direct consequence of the Bolzano-Weierstrass Theorem.

(b) In this part of the question we will show that x_n is not convergent, though. Suppose $\lim_{n\to\infty} \cos(n) = a$ for some real number a. Using a trig identity for $\cos(n+1)$ and considering $\lim_{n\to\infty}(\cos(n+1) - \cos(n))$, show that

$$\frac{a(\cos(1) - 1)}{\sin(1)} = \lim_{n\to\infty} \sin(n).$$

But then use the sequence $\lim_{n\to\infty}(\sin(n+1) - \sin(n))$ to deduce that $a = 0$, so $\lim_{n\to\infty} \cos(n) = \lim_{n\to\infty} \sin(n) = 0$. But this is a contradiction. Explain why to conclude the proof.

Solution: The addition formula for cos implies that $\cos(n+1) = \cos(n)\cos(1) - \sin(n)\sin(1)$. If we assume that $\lim_{n\to\infty} \cos(n) = a$, then using parts of the "Big Theorem" and rearranging algebraically, we see

$$\begin{aligned} 0 &= \lim_{n\to\infty} (\cos(n+1) - \cos(n)) \\ &= \lim_{n\to\infty} \cos(n)(\cos(1) - 1) - \lim_{n\to\infty} \sin(n)\sin(1) \\ &= a(\cos(1) - 1) - \sin(1) \lim_{n\to\infty} \sin(n). \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \sin(n) = \frac{a(\cos(1) - 1)}{\sin(1)}. \tag{1}$$

as claimed. Now the addition formula for sin shows

$$\sin(n+1) = \sin(n)\cos(1) + \cos(n)\sin(1)$$

Taking the limit as $n \to \infty$ on both sides and substituting from (1), we get:

$$\frac{a(\cos(1) - 1)}{\sin(1)} = \frac{a(\cos(1) - 1)}{\sin(1)} \cos(1) + a\sin(1),$$

so

$$a(\cos(1)-1) = a(\cos(1)-1)\cos(1) + a\sin^2(1),$$

and hence (because $\cos^2(1)+\sin^2(1)=1$),

$$a(\cos(1)-1) = a(1-\cos(1)).$$

The only way this can be true is if $a=0$. But then this implies that

$$\lim_{n\to\infty}\sin(n) = 0 = \lim_{n\to\infty}\cos(n).$$

But that is impossible because it would say $\lim_{n\to\infty}\cos^2(n)+\sin^2(n)=0$ by parts (a) and (b) of the "Big Theorem." However, we know by the basic identity for the trigonometric functions that

$$\cos^2(n)+\sin^2(n)=1$$

for all $n\geq 1$. Hence $\lim_{n\to\infty}\cos^2(n)+\sin^2(n)=1$ as well if the two limits exist. This contradiction shows that $\lim_{n\to\infty}\cos(n)$ cannot exist.

3. A *cluster point* of a sequence x_n is a limit of a convergent subsequence x_{n_k}. (See question 1 on the A section for examples of sequences with several different cluster points.) Show that if a_m is a convergent sequence of cluster points of a given sequence x_n, then $a=\lim_{m\to\infty}a_m$ is also a cluster point of the x_n sequence.

Solution: We let a_m be a convergent sequence of cluster points of x_n with $a=\lim_{m\to\infty}a_m$. Since each a_m is a cluster point of the x_n sequence, there is a subsequence of x_n converging to a_m. From the subsequence converging to a_1, select any x_{n_1} with $|x_{n_1}-a_1|<1$. Then from the subsequence converging to a_2, select any x_{n_2} with $n_2>n_1$ and $|x_{n_2}-a_2|<\frac{1}{2}$, then from the subsequence converging to a_3, select x_{n_3} with $|x_{n_3}-a_3|<\frac{1}{3}$ and $n_3>n_2$. By an induction argument, we can always continue this process since for any $\ell\geq 1$, there are infinitely many index values n_ℓ for which $|x_{n_\ell}-a_\ell|<\frac{1}{\ell}$ (all the indices giving terms in the subsequence converging to a_ℓ that are distance $\frac{1}{\ell}$ or less from a_ℓ). In this way we get a subsequence $\{x_{n_\ell}\}$ (indexed by $\ell\in\mathbf{N}$) with n_ℓ strictly increasing and $|x_{n_\ell}-a_\ell|<\frac{1}{\ell}$ for all $\ell\geq 1$. We claim that the subsequence x_{n_ℓ} converges to a. To see this note that given any $\varepsilon>0$, there exists $\ell_0\in\mathbf{N}$ such that $\frac{1}{\ell}<\frac{\varepsilon}{2}$ and $|a_\ell-a|<\frac{\varepsilon}{2}$ for all $\ell\geq\ell_0$. Then by the triangle inequality, for all $\ell\geq\ell_0$,

$$|x_{n_\ell}-a| = |x_{n_\ell}-a_\ell+a_\ell-a| \leq |x_{n_\ell}-a_\ell|+|a_\ell-a| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

This shows that the subsequence $\{x_{n_\ell}\}$ converges to a, so a is also a cluster point of the $\{x_n\}$ sequence.