MATH 242 – Principles of Analysis
Solutions for Problem Set 3 – due: Feb. 14

'A' Section

1. A set B is said to be *finite* if there is some $n \in \mathbf{N}$ (the number of elements in B), and a one-to-one and onto mapping $f : \{1, 2, \ldots, n\} \to B$. (Intuitively, we think that $f(1) = b_1, f(2) = b_2, \ldots$ "counts through" all the elements of B one at a time without repetitions and without missing any elements in B.) For each of the following sets, either show B is finite by determining the n and constructing a mapping f as above, or say why no such mapping exists.
   a. $B = \{r = p/q \in \mathbf{Q} \mid 1 \le q \le 4 \text{ and } 0 < r < 1\}$
   Solution: This is a finite set containing exactly 5 elements:
   $$B = \{1/2, 1/3, 2/3, 1/4, 3/4\}$$
   We can construct a 1-1 onto mapping from $\{1, 2, 3, 4, 5\}$ to this B from the order they are listed here: $f(1) = 1/2$, $f(2) = 1/3$, $f(3) = 2/3$, $f(4) = 1/4$, $f(5) = 3/4$.
   b. $B = \{r = p/q \in \mathbf{Q} \mid 0 < r < 1\}$
   Solution: B is *not* a finite set because, for instance, it contains all of the $\frac{1}{n}$ for $n \in \mathbf{N}$.
   c. $B = \{n \in \mathbf{Z} \mid |n| \le 10^{14}\}$
   Solution: B is finite with $n = 2 \times 10^{14} + 1$ elements. $f(k) = -2 \times 10^4 - 1 + k$ defines a 1-1 and onto mapping from $\{1, 2, \ldots, n\}$ to B.
2. Which of the following sequences converge to 0? Explain your answers, but you do not need to provide complete formal proofs of your assertions.
   a. $\{x_n\}$, where
   $$x_n = \begin{cases} 2^n & \text{if } n \le 1000 \\ 2^{-n} & \text{if } n > 1000 \end{cases}$$
   Solution: Intuitively, this sequence should converge to 0 since although the first part of it, for $n \le 1000$ grows very rapidly and reaches a huge value $2^{1000} \doteq 10^{300}$, once $n > 1000$, the terms rapidly decrease to 0.
   b. $\{y_n\}$, where
   $$y_n = \begin{cases} 1 & \text{if } n \text{ is evenly divisible by } 100 \\ \frac{1}{n} & \text{if } n \text{ is not evenly divisible by } 100 \end{cases}$$

Solution: There are arbitrarily large n that are evenly divisible by 100. So there are y_n for n arbitarily large such that $y_n = 1$. (Formally, there exists $\varepsilon > 0$, like $\varepsilon = 1/2$, such that for all n_0, $|y_n - 0| > 1/2$ for some $n \geq n_0$.)

c. $\{z_n\}$, where

$$z_n = \begin{cases} n & \text{if } n \text{ is a Mersenne prime number} \\ \frac{(-1)^n}{n^2} & \text{if } n \text{ is not a Mersenne prime number} \end{cases}$$

(look these up on Wikipedia and read about them)

Solution: The question of whether there are infinitely many Mersenne prime numbers (i.e. primes of the form $M = 2^p - 1$ where p is a prime), is a famous unsolved problem. It is not currently known whether there are infinitely many such primes or not. (As of January 2014, there are 48 of them known.) Hence *we don't know whether this sequence converges or not*! If there are infinitely many Mersenne primes, the situation is like that in b. If not, that is, if there are only finitely many Mersenne primes, then the sequence does converge to 0.

3. Let $f(x) = [x]$ be the greatest integer function, defined as $[x] =$ the greatest integer $\leq x$.

a. If $x_n \to a$, does it follow that $[x_n] \to [a]$? Prove or give a counterexample.

Solution: This is false because, for instance if $x_n = 1 - 1/n$, then $0 \leq x_n < 1$ for all $n \geq 1$, so $[x_n] = 0$ for all n. But $x_n \to a = 1$, and $[a] = [1] = 1$.

b. If $[x_n] \to [a]$, does it follow that $x_n \to a$? Prove or give a counterexample.

Solution: This is also false. Here's a counterexample: Let $x_n = \frac{1}{2}$ for all n (a constant sequence). Then $[x_n] = 0$ for all n and $[x_n] \to [0]$. But $x_n \to 1/2 \neq 0$.

'B' Section

1.

a. Prove that $\sqrt{3}$ is an irrational number.

Solution: Suppose on the contrary that $\sqrt{3} = \frac{m}{n}$ where m, n are integers. We may assume m, n have no common factors (by cancelling any common factors between the numerator and denominator of the fraction). Squaring both sides and clearing denominators, we get $3n^2 = m^2$. Since 3 divides m^2 evenly, 3 must also divide m (this follows because 3 is a prime number). In other words $m = 3k$ for some integer k. But then $3n^2 = 9k^2$ so $n^2 = 3k^2$. Now, we repeat the same reasoning to claim that 3 must divide n as well. Since we assumed m, n had no common factors, we have reached a contradiction. There can be no integers m, n that satisfy the original equation $\sqrt{3} = \frac{m}{n}$. Therefore $\sqrt{3}$ is irrational.

Alternative method: The equation $3n^2 = m^2$ also implies that both n and m must be odd (since they must have the same parity and they cannot both be even since then the fraction m/n would not be in lowest terms). If $n = 2\ell + 1$ and $m = 2k+1$, we get $3(4\ell^2 + 4\ell + 1) = 4k^2 + 4k + 1$, so $6\ell^2 + 6\ell + 1 = 2k^2 + 2k$. But notice this also gives a contradiction since the left side of the last equation must be odd, and the right side must be even.

b. If $r \neq 0$ and s are rational numbers, show that $r\sqrt{3} + s$ is also an irrational number. (Hint: Suppose not and derive a contradiction.)

Solution: Suppose $r\sqrt{3} + s = x$, where x is rational. Then $\sqrt{3} = \frac{x-s}{r}$ must also be rational ($\mathbf{Q}$ is closed under sums and quotients). But that contradicts part a. So $r\sqrt{3} + s$ is also irrational.

c. If $x = r\sqrt{3} + s$ and $x' = r'\sqrt{3} + s'$ are two numbers as in part b, what can be said about $x + x'$ and xx'? Are they necessarily irrational too?

Solution: No, the sum and product are not necessarily irrational. For instance if $x = \sqrt{3}$ and $x' = -\sqrt{3}$, then $x + x' = 0$ and $xx' = -3$. Both of those *are rational.*

2. Let A and B be two nonempty sets of real numbers.

a. Assume that $x \leq y$ for all $x \in A$ and $y \in B$. Show that lub A and glb B must exist.

Solution: Let $y \in B$ (which exists because we assume B is nonempty). By the given information, $x \leq y$ for every $x \in A$. Therefore, y is an upper bound for A. By the LUB axiom for $\mathbf{R}$, A has a least upper bound in $\mathbf{R}$. Similarly every element of A is a lower bound for B, so glb B exists by the result of Corollary 1.5.11 in our text.

b. Under the same assumptions as part a, show that lub $A \leq$ glb B.

Solution: Suppose on the contrary that lub $A >$ glb B. Then glb B is not an upper bound for A, so there exists $x \in A$ with glb $B < x \leq$ lub A. But then since x is not a lower bound for B, we also have $y \in B$ such that glb $B \leq y < x$. This contradicts the assumption that $x \leq y$ for all x in A and all y in B. So it must be true that lub $A \leq$ glb B

c. Now assume that A and B are bounded. Is it true that lub $A \leq$ glb B implies that $x \leq y$ for all $x \in A$ and $y \in B$? Prove or give a counterexample.

Solution: This is true. Every $x \in A$ satisfies $x \leq \text{lub}(A)$ and every $y \in B$ satisfies $\text{glb}(B) \leq y$. But then under this assumption, $x \leq \text{lub}(A) \leq \text{glb}(B) \leq y$. So $x \leq y$ for all $x \in A$ and $y \in B$.

3. Let A be a bounded set of real numbers and let $B = \{kx \mid x \in A\}$, where $k < 0$ is a strictly negative number. Show that B is also bounded. Then, determine formulas for computing lub B and glb B in terms of lub A and glb A, and prove your assertions.

Solution: Since A is bounded (i.e. bounded above and below), there exist real numbers ℓ, u such that $\ell \leq x \leq u$ for all $x \in A$. Since k is negative, this implies $k\ell \geq kx \geq ku$. But then B is bounded too, since ku is a lower bound and $k\ell$ is an upper bound for B. Noting the reversal of the inequalities that occurred here, we claim that

(a) $\text{lub}(B) = k \cdot \text{glb}(A)$, and
(b) $\text{glb}(B) = k \cdot \text{lub}(A)$.

To prove (a), write $m = \text{glb}(A)$. By the definition, this means first that $x \geq m$ for all $x \in A$. But then $kx \leq km$, and hence km is an upper bound for B. Next we assume that u is any other upper bound for B, so $kx \leq u$ for all $x \in A$. But this implies $x \geq \frac{u}{k}$ for all $x \in A$. So since m is the greatest lower bound for A, we have $m \geq \frac{u}{k}$. But that implies $km \leq u$. Therefore, $km = \text{lub}(B)$. The proof of (b) is similar: write $M = \text{lub}(A)$. By the definition, this means first that $x \leq M$ for all $x \in A$. But then $kx \geq kM$, and hence kM is a lower bound for B. Next we assume that ℓ is any other lower bound for B, so $kx \geq \ell$ for all $x \in A$. But this implies $x \leq \frac{\ell}{k}$ for all $x \in A$. So since M is the least upper bound for A, we have $M \leq \frac{\ell}{k}$. But that implies $kM \geq \ell$. Therefore, $kM = \text{glb}(B)$.

4. Determine whether each of the following sequences converge and prove your assertions using the ε, n_0 definition of convergence.

a. $x_n = \frac{3n^2}{n^2+5}$

Solution: This sequence converges to $a = 3$. Proof: Let $\varepsilon > 0$. Since $\mathbf{N}$ is not bounded in $\mathbf{R}$, no matter how big $\sqrt{\frac{15}{\varepsilon}}$ is, there exist $n_0 > \sqrt{\frac{15}{\varepsilon}}$ in $\mathbf{N}$, and for any such n_0, $n_0^2 > \frac{15}{\varepsilon}$ (since the squaring function is increasing for positive inputs), and hence $\frac{15}{n_0^2} < \varepsilon$. Then for all $n \geq n_0$, we have

$$|x_n - 3| = \left|\frac{-15}{n^2+5}\right| < \frac{15}{n^2} \leq \frac{15}{n_0^2} < \varepsilon.$$

This shows $x_n \to 3$ as $n \to \infty$ by the definition.

b. $x_n = \frac{1}{\ln(n)}$

Solution: This sequence converges to $a = 0$. Proof: Let $\varepsilon > 0$. Since $\mathbf{N}$ is not bounded in $\mathbf{R}$, no matter how big $e^{\frac{1}{\varepsilon}}$ is, there exist $n_0 > e^{\frac{1}{\varepsilon}}$ in $\mathbf{N}$, and for any such n_0, $\ln(n_0) > \frac{1}{\varepsilon}$ (since ln is increasing), and hence $\frac{1}{\ln(n_0)} < \varepsilon$. Then for all $n \geq n_0$, we have

$$|x_n - 0| = \left|\frac{1}{\ln(n)}\right| = \frac{1}{\ln(n)} \leq \frac{1}{\ln(n_0)} < \varepsilon.$$

This shows $x_n \to 0$ as $n \to \infty$ by the definition.

c. $x_n = \cos(n\pi)$.

Solution: We have $\cos(n\pi) = 1$ if n is even and $\cos(n\pi) = -1$ if n is odd. Therefore $x_n = \cos(n\pi)$ does not converge to any $a \in \mathbf{R}$. To prove this, let us show that no matter what a is, there exist $\varepsilon > 0$ such that for all $n_0 \in \mathbf{N}$, there always exist $n \geq n_0 \in \mathbf{N}$ such that $|x_n - a| \geq \varepsilon$. For the even n we have $|x_n - a| = |1 - a|$, while for odd n, $|x_n - a| = |-1 - a| = |1 + a|$. For all $a \in \mathbf{R}$, by the triangle inequality,

$$|1 + a| + |1 - a| \geq |1 + a + 1 - a| = 2.$$

This shows that *at least one of the following is true no matter what a is:*

$$|-1 - a| = |1 + a| \geq 1 \qquad \text{or} \qquad |1 - a| \geq 1,$$

Hence if we take $\varepsilon = 1$, for instance, it will always be true that $|-1-a| = |x_n - a| \geq 1 = \varepsilon$ (n odd) or $|1 - a| = |x_n - a| \geq 1 = \varepsilon$ (n even). There are natural numbers n of both types with $n \geq n_0$ for every $n_0 \in \mathbf{N}$. Therefore the definition of convergence cannot hold with any a as the limit value.