MATH 242 – Principles of Analysis
Problem Set 3 – due: Feb. 14

*'A' Section*

1. A set $B$ is said to be *finite* if there is some $n \in \mathbf{N}$ (the number of elements in $B$), and a one-to-one and onto mapping $f : \{1, 2, \ldots, n\} \to B$. (Intuitively, we think that $f(1) = b_1, f(2) = b_2, \ldots$ "counts through" all the elements of $B$ one at a time without repetitions and without missing any elements in $B$.) For each of the following sets, either show $B$ is finite by determining the $n$ and constructing a mapping $f$ as above, or say why no such mapping exists.
   a. $B = \{r = p/q \in \mathbf{Q} \mid 1 \le q \le 4 \text{ and } 0 < r < 1\}$
   b. $B = \{r = p/q \in \mathbf{Q} \mid 0 < r < 1\}$
   c. $B = \{n \in \mathbf{Z} \mid |n| \le 10^{14}\}$
2. Which of the following sequences converge to 0? Explain your answers, but you do not need to provide complete formal proofs of your assertions.
   a. $\{x_n\}$, where
   $$x_n = \begin{cases} 2^n & \text{if } n \le 1000 \\ 2^{-n} & \text{if } n > 1000 \end{cases}$$
   b. $\{y_n\}$, where
   $$y_n = \begin{cases} 1 & \text{if } n \text{ is evenly divisible by } 100 \\ \frac{1}{n} & \text{if } n \text{ is not evenly divisible by } 100 \end{cases}$$
   c. $\{z_n\}$, where
   $$z_n = \begin{cases} n & \text{if } n \text{ is a Mersenne prime number} \\ \frac{(-1)^n}{n^2} & \text{if } n \text{ is not a Mersenne prime number} \end{cases}$$
   (look these up on Wikipedia and read about them.)
3. Let $f(x) = [x]$ be the greatest integer function, defined as $[x] =$ the greatest integer $\le x$.
   a. If $x_n \to a$, does it follow that $[x_n] \to [a]$? Prove or give a counterexample.
   b. If $[x_n] \to [a]$, does it follow that $x_n \to a$? Prove or give a counterexample.

*(over for 'B' Section problems)*

*'B' Section*

1.
   a. Prove that $\sqrt{3}$ is an irrational number.
   b. If $r \neq 0$ and $s$ are rational numbers, show that $r\sqrt{3} + s$ is also an irrational number. (Hint: Suppose not and derive a contradiction.)
   c. If $x = r\sqrt{3} + s$ and $x' = r'\sqrt{3} + s'$ are two numbers as in part b, what can be said about $x + x'$ and $xx'$? Are they necessarily irrational too?
2. Let $A$ and $B$ be two nonempty sets of real numbers.
   a. Assume that $x \leq y$ for all $x \in A$ and $y \in B$. Show that lub $A$ and glb $B$ must exist.
   b. Under the same assumptions as part a, show that lub $A \leq$ glb $B$.
   c. Now assume that $A$ and $B$ are bounded. Is it true that lub $A \leq$ glb $B$ implies that $x \leq y$ for all $x \in A$ and $y \in B$? Prove or give a counterexample.
3. Let $A$ be a bounded set of real numbers and let $B = \{kx \mid x \in A\}$, where $k < 0$ is a strictly negative number. Show that $B$ is also bounded. Then, determine formulas for computing lub $B$ and glb $B$ in terms of lub $A$ and glb $A$, and prove your assertions.
4. Determine whether each of the following sequences converge and prove your assertions using the $\varepsilon, n_0$ definition of convergence.
   a. $x_n = \frac{3n^2}{n^2+5}$
   b. $x_n = \frac{1}{\ln(n)}$
   c. $x_n = \cos(n\pi)$.