MATH 242 – Principles of Analysis
Problem Set 2 – due: Feb. 7

'A' Section

1. Let $x \in [-1, 2]$. Determine the largest and smallest values of $|x-5|$, $|x+5|$.
2. Use the binomial theorem (Theorem 1.4.1) for all parts of this problem.
   a. Expand using the binomial theorem and simplify as much as possible:
   $$(a^2 + 3b)^5.$$
   b. What is the coefficient of x^3 in the expansion of
   $$\left(x^5 + \frac{3}{x^2}\right)^3.$$
   c. Express with no summation: $\sum_{k=0}^{n} \binom{n}{k} 2^k$? Explain.
   d. Express with no summation: $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$? Explain.
3. For each of the following statements, say whether the statement is true or false. If it is false, give a counterexample; if it is true, give a short reason.
   a. A set $A \subset \mathbf{R}$ is bounded if there exists some $B > 0$ such that $|x| \le B$ for all $x \in A$.
   b. If $A, B \subset \mathbf{R}$ are bounded, then $A \cap B$ is also bounded.
   c. If $A, B \subset \mathbf{R}$ are bounded, then $D = \{x + y \mid x \in A, y \in B\}$ is also bounded.
   d. If $A, B \subset \mathbf{R}_{>0}$ are bounded, then $Q = \{y/x \mid x \in A, y \in B\}$ is also bounded.
4.
   a. Let $A = [0, 3) \cap (2, 5]$. What is $a = \text{lub } A$? What is $b = \text{glb } A$? Are $a, b \in A$?
   b. Let $B = \{x \in \mathbf{R} \mid 0 < x^2 - 2x + 1 < 1\}$. What is $a = \text{lub } B$? What is $b = \text{glb } B$? Are $a, b \in B$?

'B' Section

1. Let x, y be any real numbers.
   a. Show that $|x| - |y| \le |x - y|$ and deduce that $||x| - |y|| \le |x - y|$.
   b. Show that if $x, y > 0$, then $x < y$ is equivalent to $x^n < y^n$ for all $n \ge 1$.
   c. Show that if $0 < x < y$, then $\sqrt{y} - \sqrt{x} < \sqrt{y - x}$.
2. Let a, b be any real numbers. Define $\max(a, b)$ and $\min(a, b)$ to be the larger and smaller of the two numbers, respectively. (That is, $\max(a, b) = a$ if $a \ge b$ and $\max(a, b) = b$ if $b \ge a$. Similarly for the minimum.) Show that
   $$\max(a, b) = \frac{a + b}{2} + \frac{|a - b|}{2}$$

and

$$\min(a,b) = \frac{a+b}{2} - \frac{|a-b|}{2}.$$

(Hint: Consider the positions of a, b on the number line and then "think geometrically.")

3. Show by mathematical induction that

$$(1+x)^n \geq 1 + nx$$

for all $x > -1$ and all $n \geq 0$.

4. Find a suitable n_0 and then show by mathematical induction that $n! \geq 5^n$ for all $n \geq n_0$.