Mathematics 242 – Principles of Analysis
Information on Final Examination
May 1, 2014

General Information

- The final examination for this class will be given during the scheduled period – 8:00 to 10:30 am on Friday, May 9
- The final will be a *comprehensive* exam. It will cover all the topics from the three midterms, *plus the material about series from the past week. About* 10% *of the exam will be devoted to questions about series.* See the list of topics below for more details.
- The exam will be similar in format to the midterms (however, read the next bullet point carefully). It will be roughly 1.5 to 1.75 times as long as the midterms. In other words, it will be written to take about 1.5 hours = 90 minutes if you work steadily. But you will have the full 2.5 hour = 150 minute period to use if you need that much time.
- Since there are many interconnections between topics we studied earlier in the semester and those we studied more recently, a few questions may be posed in different ways and connect sections of the course in ways you have not seen on the previous exams. For instance, I might ask you to evaluate a limit of a sequence by using properties of functions like the natural logarithm (PS 9) and L'Hopital's Rule (PS 8). Similarly, you may need to use properties of integrals to understand whether an infinite series converges by using the Integral Test.
- If there is interest, I would be happy to arrange an evening review session during exam week – I think I'm free every evening. We can discuss this in class on May 5.

Philosophical Comments and Suggestions on How to Prepare

- The reason we give final exams in almost all mathematics classes is to encourage students to "put whole courses together" in their minds. Also, preparing for the final should help to make the ideas "stick" so you will have the material at your disposal to use in later courses.
- If you approach preparing for a final exam in the right way it can be a *real learning experience* – especially in a class like this one where almost everything we have done "fits together" in a very tight chain of logical reasoning starting with the Completeness Axiom for the real number system. Much of what we did earlier in the semester may and should make much more sense now than it may have the first time around!
- *Start reviewing now*, and do some review each day between now and May 15 (even just 1/2 hour each day will make a big difference). That way you will not be "crunched" at the end (and with any luck the ideas we have developed in this course will "stick" better!)

Topics To Be Included

0) Sets, functions

1) The real number system, rational and irrational numbers, the algebraic and order properties, least upper bounds (Axiom of Completeness)
2) Mathematical induction
3) Sequences, convergence ($\lim_{n\to\infty} x_n$ – both the ε, n_0 definition, and computing limits via the limit theorems).
4) Subsequences, The Nested Interval Theorem, and the Bolzano-Weierstrass theorem
5) Limits of functions (the ε, δ definition), the "big theorem" for function limits. The order limit theorems and "squeeze theorem" (see Theorems 3.2.8, 3.2.9 in the text for the statements).
6) Continuity, the Extreme and Intermediate Value Theorems, uniform continuity.
7) Definition and properties of the derivative, the Mean Value Theorem and its consequences
8) The definite integral, integrability, the Fundamental Theorem of Calculus
9) Infinite series – convergence and divergence, key examples such as geometric series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$-series, etc. Absolute vs. conditional convergence. Comparison, alternating series, and ratio tests for convergence.

Proofs to Know

You should be able to give precise statements of *all the definitions* listed on the course homepage and the theorems mentioned in the outline above. Also, be able to give proofs of the following:

1) Every monotone increasing sequence of real numbers that is bounded above converges.
2) The Bolzano-Weierstrass Theorem
3) The Intermediate Value Theorem (the proof of the special case we did in class)
4) The Mean Value Theorem (including the special case known as "Rolle's Theorem;" the general statement is deduced from that).

Suggested Review Problems

See review sheets for Midterm Exams 1, 2, and 3 for topics 1-8 in the list above. (Those review sheets are now reposted on the course homepage if you need another copy of any of them.)

Practice Questions

Comment: This was the final exam for my section of MATH 242 last spring. That section's final was given at 3:00pm on the final day of the exam period (i.e. the last possible time). Because students can get "burned out" in that situation with our finals schedule, I decided to take it easy and I would characterize this as a *very* straight-forward exam. Even so, as I expected, it did not turn out to be too easy for last year's class. *We don't have that schedule constraint this time and I may decide to include at least one more challenging question. So be prepared.*

I. (20) Let $A = \{x^2 - 2 : -1 < x < 2\}$ and $B = \{x : |x-1| < 4\}$. Find $\mathrm{lub}(A \cup B)$ and $\mathrm{glb}(A \cap B)$.

II.

A) (10) State the ε, n_0 definition of convergence for a sequence of real numbers.
B) (10) Identify $\lim_{n\to\infty} \frac{5n^2+1}{n^2+n+4}$.
C) (10) Show that your result in part B is correct using the definition.

III.

A) (15) Show that if x_n is a monotone increasing sequence of real numbers that is bounded above, then x_n converges to some real number.
B) (10) True/False and prove/give a reason: The sequence

$$x_n = \begin{cases} -1 + \frac{1}{n^2} & \text{if } n \text{ is a prime integer} > 1 \\ 1 - \frac{1}{n^2} & \text{if } n \text{ is not prime} \end{cases}$$

has convergent subsequences.
C) (10) True/False and prove/give a reason: The infinite "continued radical"

$$\sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \cdots}}}}}$$

represents a finite real number. (Hint: If so, that number would be the limit of a sequence defined by $x_1 = \sqrt{3}$ and $x_n = \sqrt{3 + x_{n-1}}$ for all $n \geq 2$.)

IV.
A) (15) Let

$$f(x) = \begin{cases} x + 3 & \text{if } x \text{ is a rational number} \\ -x^2 + 3 & \text{if } x \text{ is an irrational number} \end{cases}$$

Is f continuous at $x = 0$? Why or why not?
B) (25) State and prove the Intermediate Value Theorem. (You may assume as known the theorem that if f is continuous at c and $x_n \to c$ is a sequence contained in the domain of f, then $f(x_n) \to f(c)$).

V.

A) (15) Using the limit definition of the derivative, compute $f'(c)$ for $f(x) = \frac{1}{(x+3)^2}$ at a general $c \neq -3$.

B) (10) What theorem guarantees that

$$F(x) = \int_1^x \frac{1}{(t+3)^2}\, dt$$

is differentiable at $x = 2$? Exactly why does it apply here? What does it say about $F'(2)$?

C) (10) Show that if $f(x) = \frac{e^x + e^{-x}}{2}$, then for every real k, there exists a solution c of the equation $f'(c) = \frac{e^c - e^{-c}}{2} = k$.

VI. (20) In this question, you may use the summation formulas:

$$\sum_{i=1}^{n} 1 = n \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Show that $f(x) = x^2 + x$ is integrable on $[a, b] = [0, 2]$ by considering upper and lower sums for f and determine the value of $\int_0^2 x^2 + x\, dx$ by using a limit of the upper sums.

VII.

A) (5) State the definition of convergence for an infinite series $\sum_{n=1}^{\infty} a_n$.

B) (15) For which $x \in \mathbf{R}$ does the power series

$$\sum_{n=1}^{\infty} \frac{3^n x^n}{n 7^n}$$

converge? (Apply the Ratio Test and then test the series at the endpoints of your interval separately.)

Extra Credit (10) Prove that if f is continuous at c and $x_n \to c$ is a sequence contained in the domain of f, then $f(x_n) \to f(c)$.