Mathematics 242 – Principles of Analysis
Information on Exam 2
March 21, 2014

General Information

The second hour exam for the course will be given in class on Friday, March 28. This will be an in-class, closed book exam. *You may use a calculator, but no other electronic devices are allowed and no stored text or other saved or online information can be consulted during the exam.* I will be happy to hold a late afternoon or evening review session to help you prepare. Late afternoon or evening times are possible on Wednesday, March 26 or Thursday, March 27.

Topics to be Covered

The exam will cover the material we have covered since the last exam, up through and including class on Friday, March 22. (This is the content of Problem Sets 4,5,6.) (Of course, all the material from the first exam about the real number system, least upper bounds, completeness, and the technique of proof by mathematical induction is relevant here too!)

1) Sequences and convergence.
2) The Limit Theorems for sequences.
3) The Monotone Convergence Theorem for sequences and its consequences (including using it to analyze sequences defined inductively).
4) Subsequences and the Bolzano-Weierstrass Theorem.
5) The definition of the statement $\lim_{x\to c} f(x) = L$, variants such as one-sided limits, consequences, limit theorems, techniques for computing limits, including use of sequences to detect when limits do not exist, or to prove that they do.
6) The definition of continuity and its consequences (including properties of removable, jump and infinite discontinuities).

What to Expect

The format will be similar to that of the first exam. The exam will have four or five questions, each possibly with several parts. Some questions will ask for a precise statement of a definition or a theorem we have discussed. Be prepared to give careful statements of

1) The definition of convergence for a sequence, and the statement $\lim_{x\to c} f(x) = L$.
2) The Bolzano-Weierstrass Theorem.

Also know and be able to give these proofs:

1) Part (a) of the Limit Theorem for sequences (Theorem 2.2.5) (limit of a sum is the sum of the limits).

2) The Monotone Convergence Theorem for sequences (Theorem 2.3.3 in our text).
3) The proof that if f is continuous at c and $\{x_n\}$ is a sequence convering to c with x_n in the domain of f for all n, then $f(x_n)$ converges to $f(c)$ (Theorem 3.4.13 in text).

Practice Questions

Don't be concerned about the length of this list. The actual exam will be roughly the same length as Exam 1. The idea is to show the range of different types of questions and topics that might be covered.

I. Give an example of each of the following, or give a short proof that there are no such examples:

A) A convergent sequence with all strictly negative terms whose limit is 1.
B) A sequence $\{x_n\}$ such that $\{x_n^2\}$ converges but $\{x_n\}$ does not.
C) Sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim x_n = +\infty$ but $\{x_n y_n\}$ converges to a finite limit.
D) A monotone increasing sequence $\{x_n\}$ that has no convergent subsequence.
E) A continuous function $f : \mathbf{R} \to \mathbf{R}$ such that $f(x) = 0$ if $x \in \mathbf{Q}$, but $f(\pi) = 3$.
F) A function $f : [a, b] \to \mathbf{R}$ with $f(a) = 3$, $f(b) = -1$, but such that $f(x) = 0$ has no solution in $[a, b]$.
G) A function f and a c in the domain of f such that $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$, but f is not continuous at c.
H) A function that is continuous at $x = 0$ such that $f(x) \geq 0$ for all $x \neq 0$ but $f(0) = -.00000001$
I) A sequence with subsequences converging to 7 different limits (and no others).
J) A function that is continuous at c for which there is a sequence $x_n \to c$ in the domain of f with $\{f(x_n)\}$ unbounded.
K) A continuous function f on a closed interval $[a, b]$ such that $M = \text{lub}\{f(x) \mid x \in [a, b]\}$ but $M \neq f(x)$ for any $x \in [a, b]$.

II. Show using the definition that the sequence $x_n = \frac{n+1}{3n+2}$ converges to $\frac{1}{3}$.

III.

A) Show that if $x_n \to a$ and $y_n \to b$, then $x_n + y_n \to a + b$.
B) Find $\lim_{n\to\infty} \sqrt{n}(\sqrt{9n+1} - 3\sqrt{n})$.

IV. Let $\{x_n\}$ be the sequence defined by $x_1 = 1$ and $x_n = \frac{3x_{n-1}+1}{5}$ for all $n \geq 2$.

A) Using mathematical induction, show that $x_n \geq 1/2$ for all $n \geq 1$.
B) Using mathematical induction, show that $\{x_n\}$ is monotone decreasing.
C) Use part B to show that the sequence x_n converges to $1/2$.
D) An alternate proof that this sequence converges: Using mathematical induction, show that $\left|x_n - \frac{1}{2}\right| = \frac{1}{2} \cdot \left(\frac{3}{5}\right)^{n-1}$ for all $n \geq 1$ and use that to show that $x_n \to 1/2$.

V. Prove the following statements using the $\varepsilon - \delta$ definition of functional limits:

A) $\lim_{x\to 3} x^2 - 2x + 4 = 7$.

B) $\lim_{x\to e}[x] = 2$ ($[x]$ is the greatest integer function)
C) $\lim_{x\to 2} \frac{1}{x^2} = \frac{1}{4}$.

VI. Let $f(x) = x^7 - 5x + 1$. Prove that $f(x)$ is continuous at all $c \in \mathbf{R}$.

More Challenging Problems

You should expect about 10 of the 100 points possible on the exam to be a problem that is more challenging (to separate the A's from the A-'s). I will also give an Extra Credit problem. This is indicative of the level for those components on this exam. VII. In this problem, we say a sequence $\{x_n\}$ is pC ("pseudo-Cauchy") if for every $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $|x_n - x_{n+1}| < \varepsilon$ for all $n \geq n_0$ (the "real Cauchy" sequences are the ones for which there exists n_0 such that $|x_n - x_m| < \varepsilon$ for all $m, n \geq n_0$–be sure you understand the difference!)
A) Show that if $\{x_n\}$ is convergent, then $\{x_n\}$ is pC.
B) Consider the sequence defined by $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Show that $\{x_n\}$ is pC.
C) Consider the subsequence of the sequence in part B for $n_k = 2^k$. Then $x_{n_1} = 1 + \frac{1}{2}$ $x_{n_2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, and so forth. Show that $x_{n_k} > 1 + \frac{k}{2}$ for all $k \geq 1$.
D) From the result in part C, is the sequence in part B convergent? Explain.

VIII. In all parts of this problem, $\{x_n\}$ is a bounded sequence. Let S be the set of all real numbers s such that there exists a subsequence $\{x_{n_k}\}$ converging to s. (S is the set of *cluster points* of the sequence, as defined in Problem Set 5.)

A) Show that S is bounded above and below.

The number $t = \mathrm{lub}(S)$ is sometimes called the *limit superior* of the sequence $\{x_n\}$, written

$$t = \limsup\ x_n.$$

B) What is $\limsup\ (-1)^n + \frac{1}{n}$?
C) Show that if $\varepsilon > 0$ and $t = \limsup\ x_n$, then there are only finitely many terms of the sequence x_n with $x_n > t + \varepsilon$.