4/10 The Cantor Set and Function

Recall we saw that if f is differentiable on (a,b) and $f'(c)=0$ for all $c \in (a,b)$, then f is constant.

Question: Can we "relax" the hypotheses here?

Answer: Yes to some extent, but not as far as you might think! (There are some truly strange functions out there!)

In particular, today, we want to construct a famous example of a function $f:[0,1]\to[0,1]$ such that:

(1) f is continuous on $[0,1]$

(2) f is differentiable "almost everywhere" on $[0,1]$, with $f'(c)=0$ whenever $f'(c)$ exists.

will come back to this later

(3) $f(0)=0$, $f(1)=1$ (so nonconstant!)

The construction depends on another famous example of a subset of $\mathbb{R}$ with unexpected properties, called the Cantor set.

Georg Cantor

Here's the Cantor set.

- Start with $C_0 = [0,1]$
- $C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$ (i.e. remove $(\frac{1}{3},\frac{2}{3})$ from C_0)
- $C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$ (i.e. ...)

$\vdots$

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def C, the Cantor set is

$$C = \bigcap_{n=1}^{\infty} C_n .$$

Observations:

(1) $C \neq \emptyset$, since for instance $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \dots$ are in C.

geom. series

(2) $$\frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + \cdots = \sum_{k=0}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^k$$

$$= \frac{1/3}{1 - 2/3} = 1$$

So: The total length of the (open) intervals removed from $[0,1]$ to get C is 1 (!)

(3) C is "as far as possible from empty". In fact, C is uncountably infinite.

We can represent each real in $[0,1]$ as a ternary (base 3) fraction, eg.

$$\frac{1}{3} = (.1)_3 = (.0\overline{2})_3$$

$$\frac{2}{3} = (.2)_3 = (.2\overline{0})_3$$

$$\frac{1}{2} = (.\overline{1})_3 \qquad \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots \quad \frac{1/3}{1 - 1/3} = \frac{1}{2} \checkmark$$

$$\begin{array}{r} .1 \\ 2\overline{)1.0} \\ 2 \\ \hline 1 \end{array}$$

$$1 = (.\overline{2})_3 \qquad \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots = \frac{2/3}{1 - 1/3} = 1 \checkmark$$

all x here: $x = (.1 \cdots)_3$

Note: $[0, \frac{1}{9}](\frac{1}{9}, \frac{2}{9})[\frac{2}{9}, \frac{1}{3}](\frac{1}{3}, \frac{2}{3})[\frac{2}{3}, \frac{7}{9}](\frac{7}{9}, \frac{8}{9})[\frac{8}{9}, 1]$

all x here have $\{x = (.01 \cdots)_3$

all x here $x = .21 \cdots$

etc. so

$$C = \left\{ x \in [0,1] \;\middle|\; x \text{ has a } \underline{\text{ternary}} \text{ expansion using only the digits } 0, 2 \right\}$$

(4) $\exists \quad f : C \longrightarrow [0,1]$ onto ($C \subset [0,1]$)

$$.202220 \cdots \longmapsto (.10110 \cdots)_2$$

$\Rightarrow$ also exists $\bar{f} : C \longrightarrow [0,1]$ 1-1 onto, and hence C is uncountable infinite.

Cantor function

$$f(x) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{a_n/2}{2^n} & \text{if } x \in C \quad x = (.a_1 a_2 \cdots)_3 \\[2ex] \displaystyle\sum_{n=1}^{k-1} \frac{a_n/2}{2^n} + \frac{1}{2^k} & \text{if } x \notin C \end{cases}$$

$x = .a_1 a_2 \cdots 1 \cdots$

1st 1 in k-th digit

so e.g. $f(x) = \frac{1}{2}$ all $x \in (\frac{1}{3}, \frac{2}{3})$

$f(x) = \frac{1}{4}$ all $x \in (\frac{1}{9}, \frac{2}{9})$

$f(x) = \frac{3}{4}$ all $x \in (\frac{7}{9}, \frac{8}{9})$

$\Rightarrow f$ is constant (hence diff'able, with derivative zero) on each of the intervals removed from $[0,1]$ to get C.

"Devil's Staircase" continuity on $[0,1]$.