Mathematics 242 – Principles of Analysis
Problem Set 9 – **Due:** Friday, April 19

'A' Section

1. Let $f(x) = x^2 + 3x + 3$ on $[1, 2]$.

(a) Show that f is integrable on $[1, 2]$ directly using the definition (that is, do not use any general theorems giving criteria for integrability). Hints: Use regular partitions of $[1, 2]$, and the summation rules

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ and } \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

(b) Determine the value of $\int_0^2 x^2 + 3x + 3 \, dx$.

2. Explain why the following inequalities must be true without evaluating the integrals involved:

(a)

$$0 < \int_0^{\pi/2} \frac{\sin(x) + \cos(x)}{x^3 + 1} \, dx < \frac{\pi}{\sqrt{2}}$$

(b)

$$\frac{1}{2} < \int_0^1 \frac{1 + x - x^2}{1 + \tan\left(\frac{\pi x}{4}\right)} \, dx < \frac{5}{4}$$

3. Let $F(x) = \int_0^x f(t) \, dt$ where

$$f(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ t + 1 & \text{if } 1 < t \leq 2 \end{cases}$$

(a) Find an explicit formula for $F(x)$ valid for all $0 \leq x \leq 2$.
(b) Is F differentiable at $x = 1$? Why or why not? Does this contradict the first part of the FTC?

4. Find the following derivatives using the FTC:

(a) $\frac{d}{dx} \int_0^x \frac{\sin(t)}{t} \, dt$

(b) $\frac{d}{dx} \int_{-x^2}^{x^3} e^{-u^2} \, du$

'B' Section

1. Let f be integrable on the interval $[a, b]$, and assume $\int_a^b f(x)\,dx > 0$. Show that there exist $k > 0$ and an interval $[c, d] \subseteq [a, b]$ such that $f(x) > k > 0$ for all $x \in [c, d]$.

2. Let f be continuous on $[a, b]$. Show that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

(Hint: Look at Theorem 5.2.5 b.)

3. Logarithm and Exponential. In this problem, we will construct the natural logarithm and and the exponential function e^x "from scratch," without relying on intuition about exponentials (as you probably did in calculus). We start by considering the function

$$L(x) = \int_1^x \frac{1}{t}\,dt, \tag{1}$$

for $x > 0$. Note that $\frac{1}{t}$ is continuous on $(0, +\infty)$, hence the FTC applies to show that L is a differentiable function. From calculus you probably recognize that $L(x) = \ln(x)$. We want to show directly that this makes sense and use this function to construct the inverse function $E(x) = L^{-1}(x)$ which is callled $E(x) = e^x$. Why would we proceed "backwards" like this? The issue is that, while $a^{m/n} = (a^{1/n})^m$ makes immediate sense for any positive $a \in \mathbf{R}$ and any rational exponent, what does a^x actually mean if $x \notin \mathbf{Q}$? Instead of trying to define that directly, we will take an end run around the question.

(a) We claim that the function L "has the right property to be a logarithm" – namely that $L(x \cdot x') = L(x) + L(x')$ for all $x, x' > 0$. Prove this by showing that for any constant $x' > 0$, the function $L(xx')$ also satisfies $\frac{d}{dx}L(xx') = \frac{1}{x}$ for all $x > 0$. Deduce that $L(xx') = L(x) + c$ for some constant c, then determine c by substituting an appropriate value for x.

(b) Show that $L(x)$ is strictly increasing for $x > 0$, hence is a 1-1 function on the domain $(0, +\infty)$. Hence L has an inverse function $E : \mathbf{R} \to \{x \in \mathbf{R} \mid x > 0\}$.

(c) Show that the inverse function E satisfies the equation $E(x + x') = E(x) \cdot E(x')$, hence E looks like an exponential function.

(d) We define $x = e$ as the unique solution of the equation $L(x) = 1$. Show that $x = e^{m/n}$ satisfies $L(x) = m/n$ for all rational numbers $x = m/n$.

(e) Show that the inverse function $E(x)$ of $L(x)$ is differentiable and satisfies $E'(x) = E(x)$. Hint: Look at Section 4.4. What happens if you differentiate on both sides of the equation $L(E(x)) = x$?

(f) Show that

$$E(x) = e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

for all $x \in \mathbf{R}$. Hint: Take logarithm and use L'Hopital's Rule.