Mathematics 242 – Principles of Analysis
Problem Set 7 – **due:** Friday, April 12

'A' Section

1. Let $f(x) = \frac{15x}{x^4+x^2+1}$. Using the Intermediate Value Theorem,
A) Show: For all $k \in [-5, 5]$, there exist $c \in [-1, 1]$ such that $f(c) = k$.
B) Show: For all k with $0 < k < 5$, there exist some $c \in (1, \infty)$ such that $f(c) = k$.
C) Show that if m satisfies $0 < m < 15$, then $f(x) = mx$ has two real solutions other than $x = 0$.

2. Show that there are at least three real solutions of the equation $\sin(x) + 2\cos(x) = x/2$. Hint: Look at the values of $g(x) = \sin(x) + 2\cos(x) - x/2$ at "nice" multiples of $\frac{\pi}{2}$.

3. Using the definition of the derivative, find the value of $f'(c)$, or say why f is not differentiable at $x = c$:

A) $f(x) = x^3 + 2x + 1$ at $c = 2$.
B) $f(x) = \sin(|x|)$ at $c = 0$. Hint: Look back at Problem Set 6, B 2.
C) The function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x > 1 \\ 2x - 1 & \text{if } x \leq 1 \end{cases}$$

at $c = 1$.
D) The function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{Q}^c \end{cases}$$

at $c = 0$.

4. Suppose f, g are differentiable functions with $f(g(x)) = \frac{x}{x^2+1}$ and that

$$\begin{array}{ll} g(1) = 1 & g(2) = 4 \\ g'(1) = 2 & g'(2) = -1 \end{array}$$

Determine the equation of the tangent line to the given graph at the given point.
A) $y = f(x)$ at $(1, f(1))$.
B) $y = (f \circ g)(x)$ at $(2, (f \circ g)(2))$.

'B' Section

1. Let f be continuous on $[0, 1]$ with $f(0) < 0$ and $f(1) > 1$. Suppose that g is another continuous function on $[0, 1]$ such that $g(0) \geq 0$ and $g(1) \leq 1$. Show that there exists some $c \in (0, 1)$ such that $f(x) = g(x)$.

2. Let f be continuous on $[a, b]$ with $f(a) < k < f(b)$. Here is a variation on our proof of the Intermediate Value Theorem.

A) Let
$$T = \{x \in [a,b] \mid f(x) > k\}.$$
Show that T has a greatest lower bound and that $f(\text{glb}(T)) = k$.

B) Will this glb(T) always be the same as the c we found in our proof of the IVT with $f(c) = k$? If so, prove they are the same; if not, give a counterexample.

3. This property deals with another property of real-valued functions of a real variable sometimes called *Lipschitz continuity*.

A) Let f be a function on an interval I with the property that there exists a strictly positive constant k such that $|f(x) - f(x')| \leq k|x - x'|$ for all $x, x' \in I$ (this is the definition of Lipschitz continuity). Show that f is uniformly continuous on I.

B) The converse of the statement in part A is not true: Show that $f(x) = x^{1/3}$ is uniformly continuous on $[-1,1]$, but there is no constant k such that $|f(x) - f(x')| \leq k|x - x'|$ for all $x, x' \in [-1,1]$. Hint: Think slopes of secant lines to the graph $y = x^{1/3}$.

4. Let f and g be differentiable on (a,c) and let $b \in (a,c)$. Assume $f(b) = g(b)$. Define a new function by
$$p(x) = \begin{cases} f(x) & \text{if } x \in (a,b) \\ g(x) & \text{if } x \in [b,c) \end{cases}$$
Show that p is differentiable on (a,c) if and only if $f'(b) = g'(b)$.