Mathematics 242 – Principles of Analysis
Problem Set 6 – **due:** Friday, March 22

'A' Section

1. Determine whether each of the following limits exists, then prove that your answers are correct using the ε, δ definition.

(a) $\lim_{x \to 3} x^2 - 4x + 1$
(b) $\lim_{x \to \frac{1}{2}} x + \frac{1}{x}$
(c) $\lim_{x \to 2} \frac{x^3-8}{x^2-4}$
(d) Let

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

and consider $\lim_{x \to 0} f(x)$.

2. Which of the functions in question 1 are *continuous* at the indicated c in the limits there? Explain.

3. True-False. For the true statements, give a short proof. For the false statements give a counterexample.

(a) If $\lim_{x \to 1} f(x) = e - \frac{27}{10}$, then there exists a $\delta > 0$ such that $f(x) > 0$ for all x with $0 < |x - 1| < \delta$.
(b) If $|f(x)| \leq x^3$ for all x, then $\lim_{x \to 2} f(x) = 8$.
(c) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by this rule:

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ -2x & \text{if } x \text{ is irrational.} \end{cases}$$

Then $\lim_{x \to 0} f(x)$ exists and equals 0.
(d) If $f(x) < g(x)$ on a deleted neighborhood of c, $\lim_{x \to c} f(x) = L$, and $\lim_{x \to c} g(x) = M$, then $L < M$.

'B' Section

1. Assume that $\lim_{x \to c} f(x) = L$.

(a) Show that there exists a constant B and $\delta > 0$ such that $|f(x)| \leq B$ for all x in the deleted neighborhood $\{x \in \mathbf{R} \mid 0 < |x - c| < \delta\}$.
(b) Using part (a), *not* the limit product rule, show that $\lim_{x \to c}(f(x))^n = L^n$ for all integers $n \geq 1$.
(c) Assume that $f(x) \geq 0$ on some deleted neighborhood of $x = c$. Show that

$$\lim_{x \to c} \sqrt{f(x)} = \sqrt{L}.$$

(*Hint:* It may help to treat the cases $L = 0$ and $L \neq 0$ separately.)

2. In this problem you will show that

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

For $0 < \theta < \frac{\pi}{2}$, the point $P = (\cos(\theta), \sin(\theta)) = (x, y)$ lies on the arc of the unit circle $x^2 + y^2 = 1$ in the first quadrant.

(a) Let $O = (0, 0)$, $Q = (\cos(\theta), 0)$, and $R = (1, 0)$. (Draw a picture!) By considering the areas of the triangle ΔOQP and the circular sector ORP, deduce that if $0 < \theta < \frac{\pi}{2}$, then $\sin(\theta)\cos(\theta) \leq \theta$. (You may use "intuitively reasonable" facts about areas such as the statement that if one plane region $\mathcal{R}$ is completely completely contained in a second region $\mathcal{S}$, then area$(\mathcal{R}) \leq$ area$(\mathcal{S})$.)

(b) Now take the tangent line to the circle at R (a vertical line), and let $S = (1, \tan(\theta))$ be the intersection of that line and the radius OP (extended). Considering the areas of the triangle ΔORS and the sector ORP as above, explain why $\theta \leq \tan(\theta)$.

(c) Combine parts (a) and (b) to deduce that if $0 < \theta < \frac{\pi}{2}$, then

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq \frac{1}{\cos(\theta)}.$$

(d) Using the one-sided form of Theorem 3.2.9 (The Limit Squeeze Theorem), show that

$$\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1.$$

(You will need to use the fact that $\cos(\theta)$ is continuous at $\theta = 0$.)

(e) Now, for $-\frac{\pi}{2} < \theta < 0$, show that $\frac{\sin(\theta)}{\theta} = \frac{\sin(|\theta|)}{|\theta|}$ and use this to see that

$$\lim_{\theta \to 0^-} \frac{\sin(\theta)}{\theta} = 1$$

as well.

(f) Finally, explain how parts (d) and (e) combine to show the statement at the start of the problem.