Mathematics 242 – Principles of Analysis
Problem Set 5
Due: March 15, 2013

'A' Section

1. For each of the following sequences, determine three different subsequences, each converging to a different limit. For each one, express your three subsequences as x_{n_k} for a suitably chosen (strictly increasing) index sequence n_k, and give an explicit formula for n_k as a function of k:

(a) $x_n = \cos\left(\frac{\pi\sqrt{n}}{2}\right)$
(b) $x_n = \frac{n}{3} - \left[\frac{n}{3}\right]$ (as usual, [ ] denotes the greatest integer function)

2. Let $x_n = n^{1/4}$. For each of the following sequences, either express that sequence as a subsequence of the sequence x_n for some explicit (strictly increasing) index sequence n_k, or say why that is impossible:

(a) $\{2, 3, 4, 5, \ldots\}$
(b) $\{\sqrt{3}, \sqrt{6}, \sqrt{9}, \sqrt{12}, \ldots\}$
(c) $\{1, 2, 4, 8, 16, 32, \ldots\}$

3. Let $x_n = \sin\left(\frac{n\pi}{4}\right)$ and $y_n = \cos\left(\frac{n\pi}{4}\right)$.

(a) Find a (strictly increasing) index sequence n_k such that both x_{n_k} and y_{n_k} converge.
(b) Find a second (strictly increasing) index sequence n_k such that both x_{n_k} and y_{n_k} diverge.
(c) Find a third (strictly increasing) index sequence n_k such that one of x_{n_k} and y_{n_k} converges and the other diverges.

'B' Section

1. (True or False) – If the statement is true give a proof; if it is false give a counterexample.

(a) If x_n is a sequence of strictly negative numbers converging to 0, then x_n has a strictly increasing subsequence x_{n_k}.
(b) If $x_n \to 0$, then x_n contains a strictly increasing subsequence or a strictly decreasing subsequence (or both).
(c) If x_n is a decreasing sequence with a bounded subsequence x_{n_k}, then x_n converges.

2. Consider the sequence $x_n = \cos(n)$ (where we think of n as an angle expressed in radians).

(a) Prove that x_n has a convergent subsequence.

(b) In this part of the question we will show that x_n is not convergent, though. Suppose $\lim_{n\to\infty} \cos(n) = a$ for some real number a. Using a trig identity for $\cos(n+1)$ and considering $\lim_{n\to\infty}(\cos(n+1) - \cos(n))$, show that

$$\frac{a(\cos(1) - 1)}{\sin(1)} = \lim_{n\to\infty} \sin(n).$$

But then use the sequence $\lim_{n\to\infty}(\sin(n+1) - \sin(n))$ to deduce that $a = 0$, so $\lim_{n\to\infty} \cos(n) = \lim_{n\to\infty} \sin(n) = 0$. But this is a contradiction. Explain why to conclude the proof.

3. A *cluster point* of a sequence x_n is a limit of a convergent subsequence x_{n_k}. (See question 1 on the A section for examples of sequences with several different cluster points.)

(a) Show that there exists a sequence x_n whose set of cluster points is all of $\mathbf{R}_+ = \{x \in \mathbf{R} \mid x > 0\}$. (Hint: Look at the solution for the Extra Credit problem on Exam 1, which shows how to get a sequence containing all the positive rational numbers. Think rational approximations to decimals to get a subsequence converging to any given positive $a \in \mathbf{R}$. However, note that you will need to be careful to produce an actual subsequence x_{n_k} with a strictly increasing index sequence n_k. In fact, this example can be "jazzed up" to get a sequence whose set of cluster points consists is all of $\mathbf{R}$!)

(b) Show that if a_m is a convergent sequence of cluster points of a given sequence x_n, then $a = \lim_{m\to\infty} a_m$ is also a cluster point of the x_n sequence.