Mathematics 242 – Principles of Analysis
Problem Set 4
Due: March 1, 2013

'A' Section

1. Using the 'Big Theorem' on limits of sequences (Theorem 2.2.5 in the text), find the limit of each of the following:

(a) $x_n = \frac{(3^n+4^{n/2})^2}{9^n}$

(b) $x_n = \sqrt{n}(\sqrt{16n+3} - 4\sqrt{n})$ (Hint: $\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$ – do you see why? Also, you may use the result of question 1 in the B section if needed.)

(c) $x_n = \frac{1}{n^4}\sum_{k=0}^{4}\binom{n}{k}\frac{1}{2^k}$. (The $\binom{n}{k}$ are the binomial coefficients as in Problem Set 2.)

2. For each of the following statements, first say whether the statement is true. Then, if the statement is true, give a reason; if it is false give a counterexample.

(a) Let $\{x_n\}$ and $\{y_n\}$ be sequences. If it is known that $\{x_n\}$ and $\{x_n + y_n\}$ converge, then $\{y_n\}$ converges too.
(b) If $\{x_n\}$ and $\{y_n\}$ both diverge, then so does $\{x_n \cdot y_n\}$.
(c) If $x_n \to a$, $y_n \to b$ and $x_n < y_n$ for all $n \in \mathbf{N}$, then $a < b$.
(d) If $x_n > 0$ and $\frac{x_{n+1}}{x_n} \leq 1$ for all $n \geq 1$, then $x_n \to 0$.

3. For each of the following sequences, first explain why the sequence is monotonic and bounded, then determine the limit:

(a) $x_{n+1} = \sqrt{6 + x_n}$, $x_1 = 1$.

(b) $x_{n+1} = \frac{4x_n+7}{10}$, $x_1 = 2$.

'B' Section

1. Let $a > 0$. Show, using the ε, n_0 definition, that if $x_n \to a$ and $x_n \geq 0$ for all n, then $\sqrt{x_n} \to \sqrt{a}$. (Hint: See part (b) of question 1 in the A section.)

2. In this problem, you will show that the sequence $x_n = \sum_{k=1}^{n} \frac{1}{k!}$ converges.

(a) First show that $\frac{1}{k!} \leq \frac{1}{2^{k-1}}$ for all integers $k \geq 1$.
(b) Show that for all $n \geq 1$,

$$1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}$$

and deduce that x_n is bounded above by 2.

(c) Use part (b) to show that x_n converges.

3. In class, we proved the Monotone Convergence Theorem for sequences, the result stated as Theorem 2.3.3 in the text. In the proof we used the Least Upper Bound Axiom in a crucial way. It is an interesting fact that the implication goes the other way too. Namely if we assume that monotone increasing sequences of reals always converge to real numbers, it follows that that every set of reals that is bounded above has a least upper bound. You will show this in this problem using the following construction. Let A be any nonempty set of real numbers that is bounded above. The proof is based on constructing two sequences x_n and y_n by this inductive procedure:

(i) Let $x_1 \in A$ be any element of A, and let y_1 be any upper bound for A.

(ii) Assuming x_k and y_k have been constructed such that $x_k \in A$ and y_k is an upper bound for A, find x_{k+1} and y_{k+1} like this: Let $m_k = \frac{x_k+y_k}{2}$. If m_k is also an upper bound for A, then set $x_{k+1} = x_k$ and $y_{k+1} = m_k$. Otherwise, there is some $x \in A$ with $m_k < x \leq y_k$, and in this case let $x_{k+1} = x$ and $y_{k+1} = y_k$.

Note that an easy induction argument shows that this construction will produce sequences such that $x_n \in A$ for all $n \geq 1$ and y_n is an upper bound for A for all $n \geq 1$.

Now prove the following statements to show that A has a least upper bound in $\mathbf{R}$ assuming the Monotone Convergence Theorem:

(a) Prove that both sequences $\{x_n\}$ and $\{y_n\}$ converge to elements of $\mathbf{R}$, using the Monotone Convergence Theorem and its Corollary.

(b) Show that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ by showing that $|x_n - y_n| = \frac{|x_1-y_1|}{2^{n-1}}$ for all $n \geq 1$.

(c) Let α be the common limit of the x_n and y_n sequences. Show that $\alpha = \text{lub}(A)$ by showing it satisfies the definition of a least upper bound.