MATH 242 – Principles of Analysis
Solutions for Problem Set 3 – due: Feb. 15

'A' Section

1. A set B is said to be *finite* if there is some $n \in \mathbf{N}$ (the number of elements in B), and a one-to-one onto mapping $f : \{1, 2, \ldots, n\} \to B$. (Intuitively, we think that $f(1) = b_1, f(2) = b_2, \ldots$ "counts through" all the elements of B one at a time without repetitions and without missing any elements in B.) For each of the following sets, either show B is finite by determining the n and constructing a mapping f as above, or say why no such mapping exists.
   a. $B = \{r = p/q \in \mathbf{Q} \mid 1 \leq q \leq 5, 0 < r < 1\}$
   Solution: The distinct elements of B are given by
   $$B = \{1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5\}$$
   Hence we have $n = 9$, and we can construct the required mapping by taking $f(1) = 1/2$, $f(2) = 1/3$, and so on up to $f(9) = 4/5$.
   b. $B = \{r = p/q \in \mathbf{Q} \mid 0 < r < 1\}$
   Solution: This set is not finite, because it contains, for instance all the elements $1/n$ for $n \geq 1$. Since $\mathbf{N}$ is not a finite set, B is not finite either.
   c. $B = \{n \in \mathbf{Z} \mid |n| \leq 10^9\}$
   Solution: This is a finite set with $n = 2 \times 10^9 + 1$ elements, since
   $$B = \{-10^9, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, 10^9\}.$$
   There are many correct mappings f. Perhaps the most useful way to write one down is to alternate back and forth between positive and negative elements like this $f(1) = 0$, $f(2) = 1$, $f(3) = -1$, $f(4) = 2$, $f(5) = -2$, etc. The general pattern is
   $$f(n) = \begin{cases} \frac{1-n}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

2. Which of the following sequences converge to 0? Explain your answers.
   a. $\{x_n\}$, where
   $$x_n = \begin{cases} e^n & \text{if } n \leq 1000 \\ e^{-n} & \text{if } n > 1000 \end{cases}$$
   Solution: This does converge to 0 since the part of the sequence starting after $n = 1000$ converges to 0. In other words, given any $\varepsilon > 0$, we can always take

$n_0 > 1000$ and then the initial segment of the sequence does not matter. Since $e^{-1} < 1$, the sequence with $e^{-n} = (e^{-1})^n$ for $n \geq 1000$ does converge to zero by a result shown in class.

b. $\{y_n\}$, where

$$y_n = \begin{cases} 1 & \text{if } n \text{ is evenly divisible by 1000} \\ \frac{1}{n} & \text{if } n \text{ is not evenly divisible by 1000} \end{cases}$$

Solution: This sequence does not converge to 0 since there are arbitrarily large numbers n that are divisible by 1000. Hence with $\varepsilon = 1$, for all n_0, there exist $n > n_0$ with $|x_n - 0| \geq \varepsilon = 1$.

c. $\{z_n\}$, where

$$z_n = \begin{cases} n & \text{if } n \text{ is a Fermat prime number (look up on Wikipedia)} \\ \frac{(-1)^n}{n^2} & \text{if } n \text{ is not a Fermat prime number} \end{cases}$$

Solution: The portion of the sequence for n equal to some Fermat prime is clearly *not tending to zero.* However, the only reasonable answer here is that *no one knows!* Fermat primes are prime numbers of the form $p = 2^{2^n}+1$, like $17 = 2^{2^2}+1$. There are only 5 known Fermat primes and all the numbers of the form $2^{2^n}+1$ after the largest known prime that have been factored successfully are composite (not prime). But there is no proof known that those five are the only ones. Hence the convergence of this sequence is an open question in mathematics(!)

3. Let $f(x) = [x]$ be the greatest integer function, defined as $[x]$ = the greatest integer $\leq x$.

   a. If $x_n \to a$, does it follow that $[x_n] \to [a]$? Prove or give a counterexample.
   This is not true. For instance, let $x_n = 1 - 1/n \to a = 1$. But $[x_n] = 0$ for all n, while $[a] = 1$.

   b. If $[x_n] \to [a]$, does it follow that $x_n \to a$? Prove or give a counterexample.
   This is not true either. For instance let $x_n = \frac{1}{2}$ for all $n \geq 1$. Then $[x_n] \to 0 = [0]$. But x_n does not converge to 0.

4.

   a. Suppose that $x_n \to e$ (the base of the natural logarithm). Show that there exists an n_0 such that $x_n < 3$ for all $n \geq n_0$.
   Solution: $e \doteq 2.71828 < 3$, so $3 - e > 0$. Applying the definition of convergence of the sequence x_n with $\varepsilon = 3 - e$, we see that there exists an n_0 such that $|x_n - e| < 3 - e$ for all $n \geq n_0$. However, $|x_n - e| < 3 - e$ is equivalent to $e - 3 < x_n - e < 3 - e$. Adding e to both sides of the second inequality here shows $x_n < 3$ for all $n \geq n_0$.

b. Suppose $x_n \to 6$ and $y_n \to 9$. Show that there exists an n_0 such that $x_n + y_n > 14$ for all $n \geq n_0$.

Solution: The idea is similar to what we did for part a. Since $x_n \to 6$, if we take $\varepsilon = 1/2$ (or any smaller positive number), there must be an $n_{0,x}$ such that $x_n > \frac{11}{2}$ for all $n \geq n_{0,x}$. Similarly, there exists an $n_{0,y}$ such that $y_n > \frac{17}{2}$ for all $n \geq n_{0,y}$. Hence if $n_0 = \max(n_{0,x}, n_{0,y})$, then $n \geq n_0$ implies $x_n + y_n > \frac{11}{2} + \frac{17}{2} = 14$.

'B' Section

1.

a. Prove that $\sqrt{5}$ is an irrational number.

Solution: Suppose $\sqrt{5} = \frac{m}{n}$ in lowest terms. Then $m^2 = 5n^2$, which shows that 5 must divide m^2. Since 5 is prime, this implies that 5 divides m, so $m = 5k$ for some integer k. But then we have $25k^2 = 5n^2$, so $5k^2 = n^2$. The same reasoning shows that n must also be divisible by 5. But that contradicts the fact that the fraction m/n was supposed to be in lowest terms.

b. If $r \neq 0$ and s are rational numbers, show that $r\sqrt{5} + s$ is an irrational number.

Solution: The set of rational numbers $\mathbf{Q}$ is closed under addition and multiplication. So if $r\sqrt{5} + s = t \in \mathbf{Q}$, then we get $\sqrt{5} = \frac{t-s}{r} \in \mathbf{Q}$. This is a contradiction to what was proved in part a.

c. If $x = r\sqrt{5} + s$ and $y = r'\sqrt{5} + s'$ are two numbers as in part b, what can be said about $x + x'$ and xx'? Are they irrational too?

Solution: Nothing can be said here since there are cases where these are rational and others where they are irrational. For instance, if $x = \sqrt{5}$ and $y = -\sqrt{5} + 1$, then $x + y = 1 \in \mathbf{Q}$. But if $x = \sqrt{5} = y$, then $x + y = 2\sqrt{5} \notin \mathbf{Q}$, by part a. Similarly, if $x = \sqrt{5} + 1$ and $y = -\sqrt{5} + 1$, then $xy = -4 \in \mathbf{Q}$. But if $x = \sqrt{5} + 1 = y$, then $xy = 6 + 2\sqrt{5} \notin \mathbf{Q}$, by part a again.

2. Let A and B be two nonempty sets of real numbers.

a. Assume that $x \leq y$ for all $x \in A$ and $y \in B$. Show that lub A and glb B must exist.

Solution: Any element of B is an upper bound for A. Hence by the lub axiom, A has a least upper bound. Similarly, any element of A is a lower bound for B, so B has a greatest lower bound.

b. Under the same assumptions as in part a, show that lub $A \leq$ glb B.

Solution: (By contradiction.) Suppose that the contrary inequality is true: $a =$ lub $A >$ glb $B = b$. By definition, that would say that the number $a =$ lub A is *not* a lower bound for B and similarly, $b = \text{glb } B$ is *not* an upper bound for A. Hence by definition, there exist $x \in A$ such that $b < x \le a$. But then x is not a lower bound for B, so there exists $y \in B$ with $b \le y < x$. But this contradicts the assumption that $x \le y$ for all $x \in A$ and $y \in B$. Hence the desired statement must be true.

c. Now assume that A, B are bounded subsets of $\mathbf{R}$. Is it true that $\text{lub } A \le \text{glb } B$ implies that $x \le y$ for all $x \in A$ and $y \in B$? Prove or give a counterexample.

Solution: This is true. Write $a = \text{lub } A$ and $b = \text{glb } B$. By definition, every $x \in A$ satisfies $x \le a$ and similarly every $y \in B$ satisfies $b \le y$. But then it follows that $x \le a \le b \le y$. Hence by transitivity, $x \le y$.

3. Let A be a bounded set of real numbers and let $B = \{kx \mid x \in A\}$, where $k < 0$ is a strictly negative number. Show that B is also bounded. Then, determine formulas for computing lub B and glb B in terms of lub A and glb B, and prove your assertions.

Solution: First, if A is bounded, there exist real numbers m, M such that $m \le x \le M$ for all $x \in A$. Then since $k < 0$, we have $km \ge kx \ge kM$. But this shows that B is also bounded (below by kM and above by km). The general statements here are that (because $k < 0$), $\text{lub } B = k \cdot \text{glb } A$ and vice versa $\text{glb } B = k \cdot \text{lub } A$. We will prove the first one of these (the other is similar; the proof consists of reversing some inequalities and changing glbs to lubs. So let $\ell = \text{glb } A$. Then $\ell \le x$ for all $x \in A$. Hence since $k < 0$, $k\ell \ge k \cdot x$ for all $x \in A$. But this shows that $k \cdot \ell$ is an upper bound for B. Now let m be any other upper bound for B, so $kx \le m$ for all $x \in A$. This implies $x \ge m/k$ for all $x \in A$ since. But then m/k is a lower bound for A. Since ℓ was the greatest lower bound for A, $m/k \le \ell$. But that implies $m \ge k\ell$. Therefore, $k\ell$ is the least upper bound for B.

4. Determine whether each of the following series converge and prove your assertions using the ε, n_0 definition of convergence.

a. $x_n = \frac{3n^2}{n^2+5}$

Solution: By rearranging algebraically, we see

$$\frac{3n^2}{n^2+5} = \frac{3}{1+5/n^2}$$

Intuitively, as $n \to \infty$, the $5/n^2$ is tending to 0, and we see the $x_n \to 3$. To prove this rigorously, we start out with our "preparation for the proof:"

$$\left|\frac{3n^2}{n^2+5} - 3\right| = \frac{15}{n^2+5} < \frac{15}{n^2}$$

To make this $< \varepsilon$ it will suffice to take $n > \sqrt{\frac{15}{\varepsilon}}$.

Formal proof: Let $\varepsilon > 0$, and let $n_0 > \sqrt{\frac{15}{\varepsilon}}$. Then for all $n \geq n_0$, we have $\frac{15}{n^2} < \varepsilon$, so

$$\left|\frac{3n^2}{n^2+5} - 3\right| = \frac{15}{n^2+5} < \frac{15}{n^2} < \varepsilon.$$

This shows that $x_n \to 3$.

b. $x_n = \frac{1}{\ln(n)}$

Solution: The numbers $\ln(n)$ increase without bound, so this sequence should be converging to 0. The "preparation for the proof" is to notice that

$$\frac{1}{\ln(n)} < \varepsilon \Leftrightarrow \ln(n) > \frac{1}{\varepsilon} \Leftrightarrow n > e^{\frac{1}{\varepsilon}}.$$

Formal proof: Let $\varepsilon > 0$ and let $n_0 > e^{\frac{1}{\varepsilon}}$. Then for all $n \geq n_0$, we have (since ln is a strictly increasing function)

$$\left|\frac{1}{\ln(n)} - 0\right| = \frac{1}{\ln(n)} < \frac{1}{\ln(e^{\frac{1}{\varepsilon}})} = \varepsilon.$$

This shows $\frac{1}{\ln(n)} \to 0$.

c. $x_n = \begin{cases} \frac{3n+1}{4n} & \text{if } n \text{ is even} \\ \frac{6n-3}{8n+1} & \text{if } n \text{ is odd} \end{cases}$

Solution: The even and odd parts of the sequence are both tending to 3/4 by rearrangements similar to what we did in part a. For the even part,

$$\left|\frac{3n+1}{4n} - \frac{3}{4}\right| = \frac{1}{4n},$$

which will be $< \varepsilon$ whenever $n > \frac{1}{4\varepsilon}$. Similarly, for the odd part,

$$\left|\frac{6n-3}{8n+1} - \frac{3}{4}\right| = \frac{15}{32n+4} < \frac{15}{32n}.$$

This will be $< \varepsilon$ whenever $n > \frac{15}{32\varepsilon}$. If we take n greater than *the larger of these bounds*, then we can use them both. That is the reason for the max below.

Formal proof: Let $\varepsilon > 0$, and let

$$n_0 > \max\left(\frac{1}{4\varepsilon}, \frac{15}{32\varepsilon}\right) = \frac{15}{32\varepsilon}.$$

Then whenever $n \geq n_0$ and n is even we have

$$\left|x_n - \frac{3}{4}\right| = \left|\frac{3n+1}{4n} - \frac{3}{4}\right| = \frac{1}{4n} < \varepsilon.$$

Similarly, when n is odd, we have

$$\left|x_n - \frac{3}{4}\right| = \left|\frac{6n-3}{8n+1} - \frac{3}{4}\right| = \frac{15}{32n+4} < \frac{15}{32n} < \varepsilon.$$

This shows that $x_n \to \frac{3}{4}$.

d. Show that the sequence $x_n = \sin\left(\frac{n\pi}{2}\right)$ does not converge to any $a \in \mathbf{R}$.

Solution: If we list out the first few terms in this sequence, we can see that the pattern will be repeating the four values $1, 0, -1, 0, \ldots$ forever. Thus it is pretty clear that the the terms are not approaching any single limit. To prove that this is not the case rigorously, we have to show that for every $a \in \mathbf{R}$, there exist $\varepsilon > 0$ such that for all n_0, there exist $n \geq n_0$ with $|x_n - a| \geq \varepsilon$. We consider the case where $a \neq -1, 0, 1$ first. In this case, let $\varepsilon = \min(|a+1|, |a|, |a-1|)$ (the shortest distance from a to any one of the three special values $-1, 0, 1$). Then in fact $|x_n - a| \geq \varepsilon$ for all n, and this shows the sequence cannot be converging to a. If $a = -1, 0, 1$, we need to argue slightly differently. Say $a = -1$ for instance (the others will be similar). If we take $\varepsilon = 1$ in this case, then notice there are infinitely many values of n for which $|x_n - (-1)| \geq 1 = \varepsilon$, namely, all the n's for which $x_n = 0$ or 1, or $n \neq 3, 7, 11, \ldots$. (In MATH 243 terms $|x_n - (-1)| \geq 1$ for all n with $n \not\equiv 3 \bmod 4$.) Hence there are such $n \geq n_0$ for any given n_0. This shows the definition of convergence does not hold with $a = -1$.