MATH 242 – Principles of Analysis
Problem Set 3 – due: Feb. 15

'A' Section

1. A set B is said to be *finite* if there is some $n \in \mathbf{N}$ (the number of elements in B), and a one-to-one onto mapping $f : \{1, 2, \ldots, n\} \to B$. (Intuitively, we think that $f(1) = b_1, f(2) = b_2, \ldots$ "counts through" all the elements of B one at a time without repetitions and without missing any elements in B.) For each of the following sets, either show B is finite by determining the n and constructing a mapping f as above, or say why no such mapping exists.
   a. $B = \{r = p/q \in \mathbf{Q} \mid 1 \leq q \leq 5 \text{ and } 0 < r < 1\}$
   b. $B = \{r = p/q \in \mathbf{Q} \mid 0 < r < 1\}$
   c. $B = \{n \in \mathbf{Z} \mid |n| \leq 10^9\}$
2. Which of the following sequences converge to 0? Explain your answers.
   a. $\{x_n\}$, where
   $$x_n = \begin{cases} e^n & \text{if } n \leq 1000 \\ e^{-n} & \text{if } n > 1000 \end{cases}$$
   b. $\{y_n\}$, where
   $$y_n = \begin{cases} 1 & \text{if } n \text{ is evenly divisible by } 1000 \\ \frac{1}{n} & \text{if } n \text{ is not evenly divisible by } 1000 \end{cases}$$
   c. $\{z_n\}$, where
   $$z_n = \begin{cases} n & \text{if } n \text{ is a Fermat prime number (look up on Wikipedia)} \\ \frac{(-1)^n}{n^2} & \text{if } n \text{ is not a Fermat prime number} \end{cases}$$
3. Let $f(x) = [x]$ be the greatest integer function, defined as $[x]$ = the greatest integer $\leq x$.
   a. If $x_n \to a$, does it follow that $[x_n] \to [a]$? Prove or give a counterexample.
   b. If $[x_n] \to [a]$, does it follow that $x_n \to a$? Prove or give a counterexample.
4.
   a. Suppose that $x_n \to e$ (the base of the natural logarithms). Explain why there exists an n_0 such that $x_n < 3$ for all $n \geq n_0$.
   b. Suppose $x_n \to 6$ and $y_n \to 9$. Explain why there exists an n_0 such that $x_n + y_n > 14$ for all $n \geq n_0$.

'B' Section

1.

   a. Prove that $\sqrt{5}$ is an irrational number.

   b. If $r \neq 0$ and s are rational numbers, show that $r\sqrt{5} + s$ is also an irrational number.

   c. If $x = r\sqrt{5} + s$ and $x' = r'\sqrt{5} + s'$ are two numbers as in part b, what can be said about $x + x'$ and xx'? Are they necessarily irrational too?

2. Let A and B be two nonempty sets of real numbers.

   a. Assume that $x \leq y$ for all $x \in A$ and $y \in B$. Show that lub A and glb B must exist.

   b. Under the same assumptions as part a, show that lub $A \leq$ glb B.

   c. Now assume that A and B are bounded. Is it true that lub $A \leq$ glb B implies that $x \leq y$ for all $x \in A$ and $y \in B$? Prove or give a counterexample.

3. Let A be a bounded set of real numbers and let $B = \{kx \mid x \in A\}$, where $k < 0$ is a strictly negative number. Show that B is also bounded. Then, determine formulas for computing lub B and glb B in terms of lub A and glb A, and prove your assertions.

4. Determine whether each of the following sequences converge and prove your assertions using the ε, n_0 definition of convergence.

   a. $x_n = \frac{3n^2}{n^2+5}$

   b. $x_n = \frac{1}{\ln(n)}$

   c. $x_n = \begin{cases} \frac{3n+1}{4n} & \text{if } n \text{ is even} \\ \frac{6n-3}{8n+1} & \text{if } n \text{ is odd} \end{cases}$

   d. $x_n = \sin(n\pi/2)$.