‘A’ Section

1. A set $B$ is said to be *finite* if there is some $n \in \mathbb{N}$ (the number of elements in $B$), and a one-to-one onto mapping $f : \{1, 2, \ldots, n\} \to B$. (Intuitively, we think that $f(1) = b_1, f(2) = b_2, \ldots$ “counts through” all the elements of $B$ one at a time without repetitions and without missing any elements in $B$.) For each of the following sets, either show $B$ is finite by determining the $n$ and constructing a mapping $f$ as above, or say why no such mapping exists.
   a. $B = \{r = p/q \in \mathbb{Q} \mid 1 \leq q \leq 5$ and $0 < r < 1\}$
   b. $B = \{r = p/q \in \mathbb{Q} \mid 0 < r < 1\}$
   c. $B = \{n \in \mathbb{Z} \mid |n| \leq 10^9\}$

2. Which of the following sequences converge to 0? Explain your answers.
   a. $\{x_n\}$, where
      $$x_n = \begin{cases} e^n & \text{if } n \leq 1000 \\ e^{-n} & \text{if } n > 1000 \end{cases}$$
   b. $\{y_n\}$, where
      $$y_n = \begin{cases} 1 & \text{if } n \text{ is evenly divisible by 1000} \\ \frac{1}{n} & \text{if } n \text{ is not evenly divisible by 1000} \end{cases}$$
   c. $\{z_n\}$, where
      $$z_n = \begin{cases} n & \text{if } n \text{ is a Fermat prime number (look up on Wikipedia)} \\ (-1)^n \frac{1}{n^2} & \text{if } n \text{ is not a Fermat prime number} \end{cases}$$

3. Let $f(x) = [x]$ be the greatest integer function, defined as $[x] = \text{the greatest integer } \leq x$.
   a. If $x_n \to a$, does it follow that $[x_n] \to [a]$? Prove or give a counterexample.
   b. If $[x_n] \to [a]$, does it follow that $x_n \to a$? Prove or give a counterexample.

4. a. Suppose that $x_n \to e$ (the base of the natural logarithms). Explain why there exists an $n_0$ such that $x_n < 3$ for all $n \geq n_0$.
   b. Suppose $x_n \to 6$ and $y_n \to 9$. Explain why there exists an $n_0$ such that $x_n + y_n > 14$ for all $n \geq n_0$. 

'B’ Section

1. 
   a. Prove that \( \sqrt{5} \) is an irrational number.
   b. If \( r \neq 0 \) and \( s \) are rational numbers, show that \( r\sqrt{5} + s \) is also an irrational number.
   c. If \( x = r\sqrt{5} + s \) and \( x' = r'\sqrt{5} + s' \) are two numbers as in part b, what can be said about \( x + x' \) and \( xx' \)? Are they necessarily irrational too?

2. Let \( A \) and \( B \) be two nonempty sets of real numbers.
   a. Assume that \( x \leq y \) for all \( x \in A \) and \( y \in B \). Show that lub \( A \) and glb \( B \) must exist.
   b. Under the same assumptions as part a, show that lub \( A \leq \) glb \( B \).
   c. Now assume that \( A \) and \( B \) are bounded. Is it true that lub \( A \leq \) glb \( B \) implies that \( x \leq y \) for all \( x \in A \) and \( y \in B \)? Prove or give a counterexample.

3. Let \( A \) be a bounded set of real numbers and let \( B = \{kx \mid x \in A\} \), where \( k < 0 \) is a strictly negative number. Show that \( B \) is also bounded. Then, determine formulas for computing lub \( B \) and glb \( B \) in terms of lub \( A \) and glb \( A \), and prove your assertions.

4. Determine whether each of the following sequences converge and prove your assertions using the \( \varepsilon, n_0 \) definition of convergence.
   a. \( x_n = \frac{3n^2}{n^2 + 5} \)
   b. \( x_n = \frac{1}{\ln(n)} \)
   c. \( x_n = \begin{cases} 
   \frac{3n+1}{4n^2} & \text{if } n \text{ is even} \\
   \frac{4n^2}{8n^2+3} & \text{if } n \text{ is odd} 
   \end{cases} \)
   d. \( x_n = \sin(n\pi/2) \).