MATH 242 – Principles of Analysis
Problem Set 2 – due: Feb. 8

'A' Section

1. Let $x \in [1,3]$. Determine the largest and smallest values of $|x-5|$, $|x+5|$, and $1/|x^2-25|$.
2. Use the binomial theorem (Theorem 1.4.1) for all parts of this problem.
   a. Expand using the binomial theorem and simplify as much as possible:
   $$(a^2 - 5b^3)^6.$$
   b. What is the coefficient of x^3 in the expansion of
   $$\left(\frac{x^4+7x}{x^2}\right)^3.$$
   c. What is $\sum_{k=0}^{n} \binom{n}{k}$? Explain.
   d. What is $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$? Explain.
3. For each of the following statements, say whether the statement is true or false. If it is false, give a counterexample; if it is true, give a short reason.
   a. A set $A \subset \mathbf{R}$ is bounded if there exists some B such that $x \le B$ for all $x \in A$.
   b. If $A, B \subset \mathbf{R}$ are bounded, then $A \cup B$ is also bounded.
   c. If $A, B \subset \mathbf{R}$ are bounded, then $D = \{x - y \mid x \in A, y \in B\}$ is also bounded.
   d. If $A, B \subset \mathbf{R}_{>0}$ are bounded, then $Q = \{x/y \mid x \in A, y \in B\}$ is also bounded.
4.
   a. Let $A = [0,4] \cap (1,5)$. What is $a = \text{lub } A$? What is $b = \text{glb } A$? Are $a, b \in A$?
   b. Let $B = \{x \in \mathbf{R} \mid 0 < x^2 - 4x + 1 < 4\}$. What is $a = \text{lub } B$? What is $b = \text{glb } B$? Are $a, b \in B$?
   c. Let $C = \{x \in \mathbf{Q} \mid x^2 < 5\}$. What is $a = \text{lub } C$? What is $b = \text{glb } C$? Are $a, b \in C$?

'B' Section

1. Let x, y be any real numbers.
   a. Show that $|x| - |y| \le |x - y|$ and deduce that $||x| - |y|| \le |x - y|$.
   b. Show that if $x, y > 0$, then $x < y$ is equivalent to $x^2 < y^2$.
   c. Show that if $0 < x < y$, then $\sqrt{y} - \sqrt{x} < \sqrt{y - x}$.

2. Let a, b be any real numbers. Define $\max(a, b)$ and $\min(a, b)$ to be the larger and smaller of the two numbers, respectively. (That is, $\max(a, b) = a$ if $a \geq b$ and $\max(a, b) = b$ if $b \geq a$. Similarly for the minimum.) Show that

$$\max(a, b) = \frac{a+b}{2} + \frac{|a-b|}{2}$$

and

$$\min(a, b) = \frac{a+b}{2} - \frac{|a-b|}{2}.$$

3. Show by mathematical induction that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

for all $n \geq 1$.

4. Show by mathematical induction that $n! \geq 2^n$ for all $n \geq 4$.