Mathematics 242 – Principles of Analysis
Solutions for Midterm Exam 3
May 3, 2013

Directions

Do all work in the blue exam booklet. There are 100 possible regular points and 10 possible Extra Credit points. Possibly useful information:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

I. Both parts of this question refer to the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$.

A) (25) Show directly, using upper and lower sums, that f is integrable on $[0, 1]$ and determine the value $\int_0^1 x^2\, dx$.

Solution: Consider a regular partition $\mathcal{P}_n$ of $[0, 1]$ with n smaller intervals. Then $\Delta x = \frac{1}{n}$ and the endpoints are $x_i = i/n$ for $i = 0, \ldots, n$. Since x^2 is increasing on the interval $[0, 1]$ we have $m_i = ((i-1)/n)^2$ and $M_i = (i/n)^2$, so

$$\begin{aligned} L(f, \mathcal{P}_n) &= \sum_{i=1}^{n} ((i-1)/n)^2 (1/n) \\ &= \frac{(n-1)n(2n-1)}{6n^3}, \text{ and} \\ U(f, \mathcal{P}_n) &= \sum_{i=1}^{n} (i/n)^2 (1/n) \\ &= \frac{n(n+1)(2n+1)}{6n^3}. \end{aligned}$$

Hence

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) = \frac{1}{n}$$

Given $\varepsilon > 0$, the difference will be $< \varepsilon$ whenever $n > \frac{1}{\varepsilon}$. So the function is integrable.

Note: The same result can be derived using the observation we made in the proof of the general result that monotone functions are integrable. Then $U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) = \frac{(f(b)-f(a))(b-a)}{n} = \frac{1}{n}$. This is certainly "cleaner!")

The value of the integral is

$$\lim_{n\to\infty} U(f, \mathcal{P}_n) = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty} \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} = \frac{1}{3}.$$

B) (5) Compute $\int_0^1 x^2\ dx$ using the Fundamental Theorem of Calculus to check your work.

Solution: An antiderivative of $f(x) = x^2$ is $F(x) = \frac{x^3}{3}$, so by part (2) of the FTC, the value is

$$\int_0^1 x^2\ dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}.$$

II. Let $f(x) = \frac{\sin(x)}{x}$ for $x \neq 0$.

A) (5) How should a value $f(0)$ be defined to make f continuous at $x = 0$? (Note: If you do not how to decide, you may "buy" the answer for 5 points so that you can do the next part.)

Solution: By results from Problem Set 7 and 8, we know

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

so we should set $f(0) = 1$ to get continuity. (Note that $\frac{\sin(x)}{x}$ is differentiable, hence continuous at all $x \neq 0$.)

B) (15) Using your value from part A for $f(0)$, use the limit definition to show that the resulting function is differentiable and find the value $f'(0)$. Say what method you are using to compute this limit.

Solution: We need to compute

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{\sin(x)}{x} - 1}{x - 0} = \lim_{x\to 0} \frac{\sin(x) - x}{x^2}.$$

This is a 0/0 indeterminate form, so we can apply L'Hopital's Rule (twice) as follows:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(x) - x}{x^2} &= \lim_{x\to 0} \frac{\cos(x) - 1}{2x} \text{ (still 0/0)}\\ &= \lim_{x\to 0} \frac{-\sin(x)}{2}\\ &= 0\end{aligned}$$

So $f'(0)$ exists and equals 0. (The graph $y = f(x)$ has a local maximum at $x = 0$.)

III.

A) (10) State the Mean Value Theorem.

Solution: Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

B) Determine

$$\text{lub}\left\{\frac{\ln(x)}{x} \mid x \in (e, +\infty)\right\}$$

and justify your answer completely using facts we have developed in this course.

Solution: Let $f(x) = \frac{\ln(x)}{x}$. By the quotient rule, we have $f'(x) = \frac{1-\ln(x)}{x^2}$. Therefore, $f'(x) > 0$ if $x < e$ and $f'(x) < 0$ for $x > e$. By one of our corollaries of the MVT, f is strictly decreasing on the given interval. That means that the value $f(e) = \frac{1}{e}$ at the left endpoint is the least upper bound of the set of values.

Note: It is not enough just to notice that $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$. To say the lub of the set of values equals $f(e)$, you must also show that f does not get larger anywhere between e and ∞. Showing f is strictly decreasing implies that.

IV. True-False. Say whether each of the following statements is true or false. For true statements, give short proofs or reasons; for false ones give reasons or counterexamples.

A) (10) $f(x) = \ln(x)$ is uniformly continuous on $(1, +\infty)$.

Solution: TRUE. $f'(x) = \frac{1}{x}$ satisfies $|f'(c)| < 1$ for all $c \in (1, +\infty)$. Therefore, as we saw on PS 8, f must be uniformly continuous on the interval $(1, +\infty)$. Here's the proof again: The MVT implies that if $1 < x_1 < x_2$, then there is some $c \in (x_1, x_2)$ where

$$|f(x_2) - f(x_1)| = |f'(c)||x_2 - x_1|.$$

But $|f'(c)| < 1$, so this shows

$$|f(x_2) - f(x_1)| < |x_2 - x_1|.$$

Hence $|f(x_2) - f(x_1)| < \varepsilon$ whenever $|x_2 - x_1| < \delta = \varepsilon$. (Or alternatively, the bound on $|f'(x)|$ shows that f is Lipschitz continuous on that interval, with Lipschitz constant $k = 1$.)

B) (10) If $A \subset \mathbf{R}$ contains no nonempty interval (a, b), then A is finite or countably infinite.

Solution: FALSE. The Cantor set is a counterexample. It contains no intervals since it is a subset of $[0, 1]$, but the intervals removed in its construction have total length 1. On the other hand, we saw in class that the Cantor set is uncountably infinite.

C) (10) Let f be continuous on $[a, b]$ and assume $f(x) > 1$ for all $x \in [a, b]$. It is possible for $\text{glb}\{f(x) \mid x \in [a, b]\}$ to equal 1.

Solution: FALSE. By the EVT, we have that f attains a minimum $m = f(c)$, for some $c \in [a, b]$. This m must equal the greatest lower bound of the set of values of f. By the given information $f(c) > 1$. Therefore the glb must also be strictly larger than 1.

Extra Credit (10) Show that if f is differentiable on an open interval I and $[a, b] \subset I$ where $f'(a) < 0$ and $f'(b) > 0$, then there must be some $c \in (a, b)$ where $f'(c) = 0$.

Solution: Since differentiability of f implies continuity of f, the EVT applies to f on the closed interval $[a, b]$ and f attains a minimum value at some $c \in [a, b]$. We claim first that $c \neq a, b$. Arguing by contradiction, suppose $f(a) \leq f(x)$ for all $x \in [a, b]$. Then we have $\frac{f(x)-f(a)}{x-a} \geq 0$ for all $x \in [a, b]$, and this implies

$$f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \geq 0.$$

But this contradicts the given information that $f'(a) < 0$. Hence the minimum is not attained at a. A similar argument shows the minimum is not attained at b either. Hence it must be attained at some $c \in (a, b)$. But now by the usual argument, we have

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \leq 0,$$

while

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0.$$

So $f'(c) = 0$.

Notes:

1) You cannot apply Rolle's theorem directly here because we are not given that $f(a) = f(b)$.
2) In addition, it is not correct to assume (even without saying so!) that f' is continuous, since that is not given either. This means, for instance, that the IVT does not necessarily apply to f' here.
3) This is the main step in a proof of a result called *Darboux's Theorem*, which says essentially that derivatives attain intermediate values, just as continuous functions do by the IVT. The general statement says for instance that if $f'(a) < k < f'(b)$ then there is some $c \in (a, b)$ with $f'(c) = k$. We can see that by applying the result here to the function $g(x) = f(x) - kx$. It is not necessarily true that the derivative function $f'(x)$ is continuous, though, so this fact is *not a consequence of the IVT*.