Mathematics 242 – Principles of Analysis
Solutions for Exam 2 – April 5, 2013

I.

A) (15) Show using the ε, n_0 definition that

$$\lim_{n\to\infty} \frac{5n+1}{3n+3} = \frac{5}{3}.$$

Solution: Given $\varepsilon > 0$, let $n_0 > \frac{4}{3\varepsilon}$. Then for all $n \geq n_0$,

$$\left|\frac{5n+1}{3n+3} - \frac{5}{3}\right| = \frac{4}{3n+3} < \frac{4}{3n} \leq \frac{4}{3n_0} < \varepsilon.$$

This shows the limit is $\frac{5}{3}$.

B) (15) Show using the ε, δ definition that

$$\lim_{x\to 1} \frac{5x+1}{3x+3} = 1.$$

Solution: Given $\varepsilon > 0$, let $\delta = \min\left(1, \frac{3\varepsilon}{2}\right)$. Then for all x in the deleted neighborhood of 1 defined by $0 < |x-1| < \delta$ we have $0 < x < 2$, so $\frac{1}{|x+1|} < 1$ and $|x-1| < \frac{3\varepsilon}{2}$. Therefore

$$\left|\frac{5x+1}{3x+3} - 1\right| = \frac{2|x-1|}{3|x+1|} = \frac{1}{|x+1|}\cdot\frac{2}{3}\cdot|x-1| < 1\cdot\frac{2}{3}\cdot\frac{3\varepsilon}{2} =< \varepsilon.$$

This shows the limit is 1.

II.

A) (20) State and prove the Monotone Convergence Theorem for sequences. (You may give the proof in the case that the sequence is monotone increasing.)

Solution: The statement is that any monotone bounded sequence of real numbers converges. If the sequence is monotone increasing, let $a = \mathrm{lub}\{x_n \mid n \in \mathbf{N}\}$. Then for all $\varepsilon > 0$, $a - \varepsilon$ is not an upper bound for $\{x_n \mid n \in \mathbf{N}\}$, so there exist n_0 such that $a - \varepsilon < x_{n_0} \leq a$. But then since $\{x_n\}$ is monotone increasing, we have $a - \varepsilon < x_{n_0} \leq x_n \leq a$ for all $n \geq n_0$. This shows $x_n \to a$, since it implies $|x_n - a| < \varepsilon$ for all $n \geq n_0$.

B) (10) Let $\{x_n\}$ be the sequence defined by $x_1 = 1$ and

$$x_{n+1} = \frac{5x_n + 3}{16}$$

for all $n \geq 1$. Does this sequence converge? Why? If it does, what is the limit?

Solution: We have $x_1 = 1$ and $x_2 = \frac{5\cdot 1+3}{16} = \frac{1}{2}$. Assuming $x_{k+1} < x_k$, it follows that

$$x_{k+2} = \frac{5x_{k+1}+3}{16} < \frac{5x_k+3}{16} = x_{k+1}.$$

Therefore, the sequence is monotone (strictly) decreasing. The terms are clearly bounded below by 0, so the sequence converges by the Monotone Convergence Theorem. The limit is found by letting $n \to \infty$ in the recurrence. If a denotes the limit then $a = \frac{5a+3}{16}$, so $a = \frac{3}{11}$.

III. (10) Let $x_n = \sin(n) - \cos(n)$ (the sequence of values of $\sin(x) - \cos(x)$ at the angles x given by a whole number $n \geq 1$ in radian measure). Show that there exists a subsequence $x_{n_k} = \sin(n_k) - \cos(n_k)$ that converges. State any "big theorems" that you are using.

Solution 1: We have

$$|x_n| = |\sin(n) - \cos(n)| \leq |\sin(n)| + |\cos(n)| \leq 2$$

for all n by the triangle inequality. This shows the sequence is bounded. Hence the Bolzano-Weierstrass theorem implies it has a convergent subsequence. (Note: This is not the tightest possible upper bound. In fact, by trig identities,

$$\sin(n) - \cos(n) = \sqrt{2}\sin\left(n - \frac{\pi}{4}\right)$$

so $|\sin(n) - \cos(n)| \leq \sqrt{2}$ is also true for all n. There are n for which $|\sin(n) - \cos(n)| > 1$, though, so that is not a valid upper bound.)

It is also possible to argue like this:

Solution 2: Since $|\sin(n)| \leq 1$ for all n, by the Bolzano-Weierstrass theorem there exists an index sequence n_k such that $\sin(n_k)$ converges to some $a \in \mathbf{R}$. There is no guarantee that $\cos(n_k)$ also converges, but we can apply the Bolzano-Weierstrass theorem again to that sequence and get a subsequence of the index sequence n_k, say n_{k_ℓ} for some strictly increasing sequence k_ℓ of k-values, such that $\cos(n_{k_\ell})$ converges to some $b \in \mathbf{R}$. But then $\sin(n_{k_\ell})$ is a subsequence of a convergent sequence, so it also converges to a. Then by the limit sum theorem, as $\ell \to \infty$,

$$\sin(n_{k_\ell}) - \cos(n_{k_\ell}) \to a - b.$$

(Note: Solution 1 is certainly more direct(!))

IV. Give an example, or give a reason why there can be no such examples:

A) (10) A function f that is continuous at all real c, with $f(x) = 3$ for all $x \in \mathbf{Q}$, but $f(\sqrt{2}) = 4$.

Solution: There are no such examples because we can find sequences $x_n \to \sqrt{2}$ where all the x_n are rational. If f is continuous at $c = \sqrt{2}$, then $f(x_n) \to f(\sqrt{2}) = 4$ but $f(x_n) = 3$ for all n if x_n is rational, so this is impossible.

B) (10) A function f that is continuous and bounded on an open interval (a, b) with $M = \text{lub}\{f(x) \mid x \in (a, b)\}$, but such that $f(x) \neq M$ for all $x \in (a, b)$.

Solution: An example is $f(x) = x$ on $(0, 1)$. We have $M = 1$, but $f(x) \neq 1$ for any $x \in (0, 1)$. (The Extreme Value Theorem does not apply on open intervals.)

C) (10) A continuous function on $[0, 1]$ with $f(0) = 4$, $f(1) = -2$, but $f(x) \neq 0$ for all $x \in (0, 1)$.
Solution: There are no such examples because the Intermediate Value Theorem says that there must be solutions of all equations $f(c) = k$ for k between $f(0) = 4$ and $f(1) = -2$, including $k = 0$.

Extra Credit. (10) A function is said to be *right-continuous* at c if the one-sided limit $\lim_{x \to c^+} f(x)$ exists and equals $f(c)$. True or False (and give a proof or a counterexample): If f is right-continuous at all $c \in [a, b]$ for some closed interval $[a, b]$, then $\{f(x) \mid x \in [a, b]\}$ is a bounded subset of $\mathbf{R}$.

Solution: This is not true. An example is

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1/x & \text{if } x < 0 \end{cases}$$

for $x \in [-1, 1]$. Then $\lim_{x \to c^+} f(x) = f(c)$ for all $c \in [-1, 1]$, but the set of values is not bounded, since $\lim_{x \to 0^-} f(x) = -\infty$.