Mathematics 242 – Principles of Analysis
Information on Final Examination
May 6, 2013

General Information

- The final examination for this class will be given during the scheduled period – 3:00 to 5:30 pm on Wednesday, May 15 (as if meeting at 8:00am wasn't bad enough, we also get *the last scheduled time* for finals!)
- The final will be a *comprehensive* exam. It will cover all the topics from the three midterms, *plus the material about series from the past week. About* 10% *of the exam will be devoted to questions about series.* See the list of topics below for more details.
- The exam will be similar in format to the midterms (however, read the next bullet point carefully). It will be roughly 1.5 to 1.75 times as long as the midterms. In other words, it will be written to take about 1.5 hours = 90 minutes if you work steadily. But you will have the full 2.5 hour = 150 minute period to use if you need that much time.
- Since there are many interconnections between topics we studied earlier in the semester and those we studied more recently, though, a few questions may be posed in different ways and connect sections of the course in ways you have not seen on the previous exams. For instance, I might ask you to evaluate a limit of a sequence by using properties of functions like the natural logarithm (PS 9) and L'Hopital's Rule (PS 8). Similarly, you may need to use properties of integrals to understand whether an infinite series converges by using the Integral Test.
- If there is interest, I would be happy to arrange an evening review session during exam week – I think I'm free every evening. We can discuss this in class on May 6.

Philosophical Comments and Suggestions on How to Prepare

- The reason we give final exams in almost all mathematics classes is to encourage students to "put whole courses together" in their minds. Also, preparing for the final should help to make the ideas "stick" so you will have the material at your disposal to use in later courses.
- If you approach preparing for a final exam in the right way it can be a *real learning experience* – especially in a class like this one where almost everything we have done "fits together" in a very tight chain of logical reasoning starting with the Completeness Axiom for the real number system. Much of what we did earlier in the semester may and should make much more sense now than it may have the first time around!
- *Start reviewing now*, and do some review each day between now and May 15 (even just 1/2 hour each day will make a big difference). That way you will not be "crunched" at the end (and with any luck the ideas we have developed in this course will "stick" better!)

Topics To Be Included

0) Sets, functions
1) The real number system, rational and irrational numbers, the algebraic and order properties, least upper bounds (Axiom of Completeness)
2) Mathematical induction
3) Sequences, convergence ($\lim_{n\to\infty} x_n$ – both the ε, n_0 definition, and computing limits via the limit theorems).
4) Subsequences, The Nested Interval Theorem, and the Bolzano-Weierstrass theorem
5) Limits of functions (the ε, δ definition), the "big theorem" for function limits. The order limit theorems and "squeeze theorem" (see Theorems 3.2.8, 3.2.9 in the text for the statements).
6) Continuity, the Extreme and Intermediate Value Theorems, uniform continuity.
7) Definition and properties of the derivative, the Mean Value Theorem and its consequences
8) The definite integral, integrability, the Fundamental Theorem of Calculus
9) Infinite series – convergence and divergence, key examples such as geometric series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$-series, etc. Absolute vs. conditional convergence. Comparison, alternating series, and ratio tests for convergence.

Proofs to Know

You should be able to give precise statements of *all the definitions* listed on the course homepage and the theorems mentioned in the outline above. Also, be able to give proofs of the following:

1) Every monotone increasing sequence of real numbers that is bounded above converges.
2) The Bolzano-Weierstrass Theorem
3) The Intermediate Value Theorem (the proof of the special case we did in class)
4) The Mean Value Theorem (including the special case known as "Rolle's Theorem;" the general statement is deduced from that).

Suggested Review Problems

See review sheets for Midterm Exams 1, 2, and 3 for topics 1-8 in the list above. (Those review sheets are now reposted on the course homepage if you need another copy.)

Practice Questions

I. Let $A = \{\cos(x) : x \in (0, 5\pi/4)\}$ and let $B = \{x : 1 < x^2 < 4\}$.

A) What are the sets $A, B, A \cap B, A \cup B, A^c$?
B) What is lub(A)? Is there a $c \in (0, 5\pi/4)$ where $f(c) = \text{lub}(A)$? Why or why not?
C) What is glb(A)? Is there a $c \in (0, 5\pi/4)$ such that $f(c) = \text{glb})(A)$? Why or why not?
B) What is the set $C = \{|x + 1| : x \in B\}$?

II.

A) State the ε, n_0 definition for convergence of a sequence.

B) Identify $L = \lim_{n\to\infty} x_n$ for the sequence

$$x_n = \frac{3n^2 + n}{n^2 + 1}$$

and prove using the definition that $\lim_{n\to\infty} x_n = L$.

C) Does $\lim_{n\to\infty} \sqrt{n}(\sqrt{n+5} - \sqrt{n})$ exist? If so, find it.

D) Show that $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$. (Use L'Hopital's Rule.)

E) Let $\{x_n\}$ be the sequence defined by $x_1 = 1$ and $x_n = \sqrt{x_{n-1} + 12}$ for $n \geq 2$. Show that $\{x_n\}$ converges and determine the limit.

III.

A) Let $\{a_n\}$ be a monotonic decreasing sequence of positive real numbers converging to 0. Let $\{s_k\}$ be the following sequence of sums of terms in the $\{a_n\}$ sequence with alternating signs:

$$s_k = a_1 - a_2 + a_3 - \cdots + (-1)^{k-1} a_k.$$

Show that for all *odd* integers $k \geq 1$, $[s_{k+3}, s_{k+2}] \subseteq [s_{k+1}, s_k]$, and deduce that $\cap_{\substack{k=1 \\ k \text{odd}}}^{\infty} [s_{k+1}, s_k] \neq \emptyset$. (Note: This includes most of the ideas used in the proof of the Alternating Series Test.)

B) Let $x_n = \sin(2\pi \cos(n))$. Show that there exists a convergent subsequence of (x_n). (Don't try to find one explicitly!)

IV.

A) Is $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$ absolutely convergent, conditionally convergent, or divergent?

B) Same question as in A for $\sum_{k=0}^{\infty} \frac{(-1)^k k^3 3^k}{k!}$.

C) For which $x \in \mathbf{R}$ does the series

$$\sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

converge? What is the radius of convergence? What happens at the endpoints of the interval?

D) Does the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^4}$ converge for some $x \in \mathbf{R}$? For all $x \in R$? Note: This *not* a power series, so you will need to figure out a method to apply here.

V.

A) Give the ε, δ definition for the statement $\lim_{x\to c} f(x) = L$.

B) Identify the limit

$$L = \lim_{x\to 0} x^{2/3} \sin(1/x)$$

and prove using the definition that $\lim_{x\to 0^+} x^{2/3} \sin(1/x) = L$.

C) The expression $x^{2/3} \sin(1/x)$ makes sense for all $x \neq 0$ in the reals, so it defines a function on the domain $(-\infty, 0) \cup (0, +\infty)$. How would we define $f(0)$ to get a function f that is continous at $x = 0$ and $f(x) = x^{2/3} \sin(1/x)$ for $x \neq 0$?

D) Is your function f from part C differentiable at 0? Why or why not?

VI.

A) Show, using the Mean Value Theorem, that if f is differentiable on an interval $I = (a,b)$ and $f'(x) \neq 0$ for all $x \in I$, then for each k the equation $f(x) = k$ has *at most one* solution with $x \in I$. (Hint: Prove the contrapositive.)

All remaining parts of this problem refer to the function

$$f(x) = \frac{32x}{x^4 + 48}$$

B) What are $f(0)$ and $f(2)$ for this function?

C) Show using the Intermediate Value Theorem that for each k with $0 < k < 1$, the equation $f(x) = k$ has *at least two* solutions $x \in \mathbf{R}$, with $x > 0$.

D) Show using part A that there are *exactly two* solutions of the equation $f(x) = k$ from part C for each k with $0 < k < 1$.

VII. In this question you may use without proof the summation formulas:

$$\sum_{i=1}^{n} 1 = n \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

A) Show that $f(x) = x^2 + x - 1$ is integrable on $[a,b] = [1,3]$ by considering upper and lower sums for f.

B) Determine the value of $\int_1^3 x^2 + x - 1\,dx$.

C) Check your answer to B by using the FTC.

VIII. True - False. For each true statement, give a short proof or reason. For each false statement give a reason or a counterexample.

A) Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with positive terms. If the partial sums s_N are bounded above by some B for all N, then $\sum_{n=1}^{\infty} a_n$ converges.

B) If f is differentiable on (a,b) and $f'(c) > 0$ for some $c \in (a,b)$, then there is an interval containing c on which f is increasing.

C) Does your answer in B change if you are also given that $f'(x)$ is continuous at all $x \in (a,b)$?

D) The function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x^2 & \text{if } x \text{ is irrational} \end{cases}$$

is *continuous* at $x = 0$.

E) There exist continuous functions $f : [0,1] \to [0,1]$ that satisfy $f(0) = 0$, $f(1) = 1$, and $f'(x)$ exists and equals 0 for all x in a union of open intervals $\cup_{n=1}^{\infty} I_n$ where $I_n = (a_n, b_n)$ with $\sum_{n=1}^{\infty}(b_n - a_n) = 1$.

F) If f is an integrable function on $[0,5]$ and $\int_0^5 f(x)\, dx = 7$, then there exists a $c \in [0,5]$ where $f(c) = \frac{7}{5}$. If so, why? If not, what extra hypothesis on f would guarantee the existence of such a c?

G) If f is continuous, non-negative function such that $\int_n^{n+1} f(x)\, dx > \frac{1}{n}$ for all natural numbers $n \geq 1$, then the infinite series $\sum_{n=1} f(n)$ diverges.

IX. Let

$$f(x) = \begin{cases} \cos(2x) & \text{if } x < 0 \\ ax^2 + bx + c & \text{if } x \geq 0 \end{cases}$$

There is exactly one set of constants a, b, c for which $f'(0)$ and $f''(0)$ both exist. Find them.