4/8 Countable + uncountable sets

- $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ is sometimes called the set of "counting numbers"

- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is also "countable"

$$-3, -2, -1, 0, 1, 2, 3$$

$$\begin{array}{ccc} 1 & \longleftrightarrow & 0 \\ 2 & \longleftrightarrow & 1 \\ 3 & \longleftrightarrow & -1 \\ 4 & \longleftrightarrow & 2 \\ 5 & \longleftrightarrow & -2 \\ & \vdots & \end{array}$$

$f : \mathbb{N} \longrightarrow \mathbb{Z}$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \frac{1-n}{2} & \text{if } n \text{ odd} \end{cases}$$

is 1-1 and onto

- def. we say $S$ is countable (or countably infinite) if there exists a 1-1 and onto mapping $f : \mathbb{N} \longrightarrow S$
- If $S \subset \mathbb{R}_+$ is countable, then so is $S \cup \{0\} \cup -S$
- $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$ is countable (!)

$\mathbb{Q}_+$:

$$\begin{array}{ccccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots \\ \frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

$\longrightarrow$ gives an onto map $f : \mathbb{N} \longrightarrow \mathbb{Q}_+$

Q: How to get 1-1 also?

- Is $\mathbb{R}$ also countable? The answer is "no" (G. Cantor, late 1800's)

- Will sketch a proof that $(0,1) \subset \mathbb{R}$ is not countable

(a) infinite decimals represent elements of $\mathbb{R}$ by LUB Axiom

Ex. $\sqrt{2} = 1.414213562 \cdots$

Construct $S = \{1, 1.4, 1.41, 1.414, 1.4142, \dots\}$
S is bounded above, hence has a LUB.
that number must equal $\sqrt{2}$

(b) Suppose we could list all of the reals in $(0,1)$ $\quad f : \mathbb{N} \to (0,1)$ 1-1 onto

n

$1 \longleftrightarrow .d_{11} d_{12} d_{13} d_{14} \cdots$ (use terminating form if it exists; i.e. $.500\cdots$ not $.49999\cdots$

$2 \longleftrightarrow .d_{21} d_{22} d_{23} d_{24} \cdots$

$3 \longleftrightarrow .d_{31} d_{32} d_{33} d_{34} \cdots$

$4 \longleftrightarrow .d_{41} d_{42} d_{43} d_{44} \cdots$

$\vdots$

Cantor "diagonal trick" make a new decimal expansion

$.e_1 e_2 e_3 e_4 \cdots$

where $e_i \neq d_{ii}$ and $e_i \neq 9$ $\Rightarrow$ e not on list!