

4/8 Countable + Uncountable sets

• $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ is sometimes called the set of "counting numbers"

• $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is also "countable"



- 1 ↔ 0
- 2 ↔ -1
- 3 ↔ -2
- 4 ↔ -3
- 5 ↔ -4
- ⋮

$f: \mathbb{N} \longrightarrow \mathbb{Z}$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \frac{1-n}{2} & \text{if } n \text{ odd} \end{cases}$$

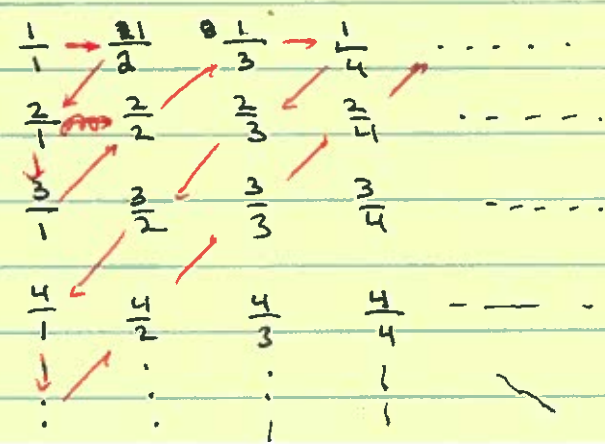
is 1-1 and onto

• def. we say S is countable (or countably infinite) if there exists a 1-1 and onto mapping $f: \mathbb{N} \longrightarrow S$

• If $S \subseteq \mathbb{R}_+$ is countable, then \mathbb{Z} is $S \cup \{0\} \cup -S$

• $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$ is countable (!)

\mathbb{Q}_+ :



→ gives an onto map

$f: \mathbb{N} \longrightarrow \mathbb{Q}_+$

Q: How to get 1-1 also?

• Is \mathbb{R} also countable? The answer is "no" (G. Cantor, late 1800's)

• Will sketch a proof that $(0, 1) \subset \mathbb{R}$ is not countable

(a) infinite decimals represent elements of \mathbb{R} by LUB Axiom

Ex. $\sqrt{2} = 1.414213562 \dots$

Construct $S = \{1, 1.4, 1.41, 1.414, 1.4142, \dots\}$

S is bounded above, hence has a LUB. That number must equal $\sqrt{2}$

(b) Suppose we could list all of the reals in $(0, 1)$ $f: \mathbb{N} \rightarrow (0, 1)$ 1-1 onto

n		
1	$\leftrightarrow \cdot d_{11} d_{12} d_{13} d_{14} \dots$	(use terminating bin if it exists; i.e. .500... not .49999...)
2	$\leftrightarrow \cdot d_{21} d_{22} d_{23} d_{24} \dots$	
3	$\leftrightarrow \cdot d_{31} d_{32} d_{33} d_{34} \dots$	
4	$\leftrightarrow \cdot d_{41} d_{42} d_{43} d_{44} \dots$	
	:	
	:	

Construct "diagonal trick" make a new decimal expansion

$\cdot e_1 e_2 e_3 e_4 \dots$

where $e_i \neq d_{ii}$ and $e_i \neq 9 \Rightarrow e$ not on list!