Mathematics 242 – Principles of Analysis
Selected Solutions – Problem Set 1, "B portion"
February 6, 2011

§1.1/

3. (a) A question like this is asking for a *specific counterexample* to the statement $(A - B) \cup B = A$ – an explicit example that shows the statement is not true. You can find a simple example where this fails whenever $A - B$ and $B - A$ are nonempty. For instance, let

$$A = \{1, 2\}, B = \{2, 3\}$$

Then $A - B = \{1\}$ so $(A - B) \cup B = \{1, 2, 3\} \neq A$.

(b) From examples like this (and/or Venn diagrams), we can see that the correct statement should be $(A - B) \cup B = A \cup B$. Here are two different "styles" of proofs of this, equally correct.

Proof 1: We will show the two inclusions $(A-B) \cup B \subset A \cup B$ and $A \cup B \subset (A-B) \cup B$. First, let $x \in (A-B) \cup B$. then $x \in A-B$ or $x \in B$ by the definition of the union. Hence by part (d) of problem 2 (part (g) of Theorem 1.1.3), $x \in A$ and $x \notin B$, or $x \in B$. This shows $x \in A$ or $x \in B$, so $x \in A \cup B$. Hence $(A-B) \cup B \subset A \cup B$. For the other inclusion, let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in B$, then $x \in (A - B) \cup B$ by definition of the union. On the other hand if $x \notin B$, then it must be true that $x \in A$, and hence $x \in A \cap B^c = A - B$ (using part (d) of problem 2 again). This shows $A \cup B \subset (A-B) \cup B$ and hence the equality we want. //

Proof 2: For this proof, we will use several parts of Theorem 1.1.3 that incorporate the sort of element-by-element arguments in Proof 1. We have

$$\begin{aligned}(A - B) \cup B &= (A \cap B^c) \cup B \text{ (by 1.1.3}(g)\text{)}\\ &= (A \cup B) \cap (B^c \cup B) \text{ (by 1.1.3}(e)\text{)}\\ &= (A \cup B) \cap X \text{ (by 1.1.3}(b)\text{)}\\ &= A \cup B \text{ (by def. of universal set)}\end{aligned}$$

§1.2/

5. Let $f : A \to B$ be a function and $C, D \subset A$.

(a) We want to show $f(C \cup D) = f(C) \cup f(D)$ (subsets of $B$), and we will do this by showing each set is contained in the other. First, let $y \in f(C \cup D)$. Then $y = f(x)$ for some $x \in C \cup D$. Hence $y = f(x)$ for some $x \in C$ or some $x \in D$. If $x \in C$, then $y = f(x) \in f(C)$; if $x \in D$, then $y = f(x) \in f(D)$. Hence $y \in f(C) \cup f(D)$. This shows $f(C \cup D) \subset f(C) \cup f(D)$. For the reverse inclusion, suppose $y \in f(C) \cup f(D)$. Then

$y \in f(C)$ or $y \in f(D)$. In the first case, $y = f(x)$ for some $x \in C$; in the second $y = f(x)$ for $x \in D$. Either way, $y = f(x)$ for some $x \in C \cup D$ since then $x \in C$ or $x \in D$. It follows that $y \in f(C \cup D)$ and $f(C) \cup f(D) \subset f(C \cup D)$.

(b) We want to show $f(C \cap D) \subset f(C) \cap f(D)$. Let $y \in f(C \cap D)$. Then $y = f(x)$ for some $x \in C \cap D$. Hence $y = f(x)$ for some $x \in C$ which also satisfies $x \in D$. Hence $y \in f(C)$ and $y \in f(D)$, so $y \in f(C) \cup f(D)$. This shows $f(C \cap D) \subset f(C) \cap f(D)$.

(c) If we think carefully about how the reverse inclusion might *fail*, then it should be clear that knowing $y \in f(C) \cap f(D)$ just says that we have some $x_1 \in C$ and some $x_2 \in D$ such that $f(x_1) = y = f(x_2)$. There's no reason why $x_1$ or $x_2$ must come from $C \cap D$. Here is an explicit example. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = x^2$. Let $C = \{1\}$ and $D = \{-1\}$. Then $f(C) = f(D) = \{1\}$, so $f(C) \cap f(D) = \{1\}$. But $C \cap D = \emptyset$, so $f(C \cap D) = \emptyset \neq f(C) \cap f(D)$.

6. We did one part of this in class(!) You should be reading your class notes in addition to working on the problem sets.

12. *Comments:* This problem is really a continuation of 5 (b) and (c) above. There will be interconnected problems on many assignments for this course; be sure you can spot them and can see what steps are necessary for the later problems, *without repeating work you already did!* Also, if a problem states an extra hypothesis (like the hypothesis that $f$ is one-to-one in part (a)), you should ask yourself: Did I use that anywhere? How did I use it?

(a) We know from 5 (b) above that $f(C \cap D) \subset f(C) \cap f(D)$ for all functions $f : A \to B$ and all subsets $C, D \subset A$. So, what must be proved for this part is that if $f$ is one-to-one, then $f(C) \cap f(D) \subset f(C \cap D)$ for all $C, D$. So let $C, D$ be subsets of $A$, and let $y \in f(C) \cap f(D)$. Then by definition we know $y \in f(C)$ and $y \in f(D)$. Hence, $y = f(x_1)$ for some $x_1 \in C$ and also (the same) $y = f(x_2)$ for some $x_2 \in D$. *Since $f$ is assumed to be one-to-one*, and $f(x_1) = y = f(x_2)$, it must be the case that $x_1 = x_2$. But then this single element is in both $C$ and $D$, hence in $C \cap D$. It follows that $y \in f(C \cap D)$, so $f(C) \cap f(D) \subset f(C \cap D)$.

(b) *Note:* A statement of the form "p if and only if q" means "if p then q, and if q then p." To prove a statement of this form, you must supply arguments for *both* implications. (This is like the standard plan for showing $A = B$, since $A \subset B$ is equivalent to "if $x \in A$, then $x \in B$" and $B \subset A$ is equivalent to "if $x \in B$, then $x \in A$").

Part (a) shows the statement "If $f$ is one-to-one, then $f(C \cap D) = f(C) \cap f(D)$ for all $C, D \subset A$." So that is already known, and what we need to show for this part is the reverse implication "If $f(C \cap D) = f(C) \cap f(D)$ for all subsets $C, D$ of $A$, then $f$ is one-to-one."

Let $x_1, x_2 \in A$ and assume $f(x_1) = f(x_2) = y \in B$. Consider the subsets $C = \{x_1\}$ and $D = \{x_2\}$ in $A$. By our hypothesis, we have

$$f(C \cap D) = f(C) \cap f(D) = \{y\}.$$

But in particular, this says $C \cap D \neq \emptyset$ (since otherwise $f(C \cap D) = \emptyset$ too). Therefore, it must be true that $x_1 = x_2$, and hence $f$ is one-to-one.

14. (a) If $ad - bc = 0$ and $a = 0$, then $bc = 0$, so $b = 0$ or $c = 0$. In the first case we have $f(x) = \frac{0x+0}{cx+d} = 0$ for all $x$ so $f$ is constant. In the second case we have $f(x) = \frac{0x+b}{0x+d}$ is constant. On the other hand, if $a \neq 0$, then we have $d = \frac{bc}{a}$ and

$$\begin{aligned} f(x) &= \frac{ax+b}{cx+\frac{bc}{a}} \\ &= \frac{a(ax+b)}{c(ax+b)} \\ &= \frac{a}{c} \end{aligned}$$

is constant again.

(b) Now assume $ad - bc \neq 0$. We can do most of this part with one algebraic calculation. Let $y = f(x)$, and try to solve for $x$ as a function of $y$. If we can do that uniquely, then $f$ is one-to-one, and we can determine the range and the formula for the inverse function all at once! The computations:

$$\begin{aligned} y &= \frac{ax+b}{cx+d} \\ (cx+d)y &= ax+b \\ x(cy-a) &= -dy+b \\ x &= \frac{-dy+b}{cy-a} \end{aligned}$$

This shows that as long as $y \neq a/c$, there is a unique $x$ such that $f(x) = y$. If $y = a/c$, then we get $ad - bc = 0$ which contradicts our hypothesis. Hence $f$ is one-to-one. The inverse function is also found from the result of this same calculation:

$$f^{-1}(x) = \frac{-dx+b}{cx-a}.$$

The domain of $f$ is

$$\text{domain}(f) = \begin{cases} \mathbf{R} - \{-d/c\} & \text{if } c \neq 0 \\ \mathbf{R} & \text{if } c = 0 \end{cases}$$

and the domain of $f^{-1}$ is

$$\text{domain}\left(f^{-1}\right) = \begin{cases} \mathbf{R} - \{a/c\} & \text{if } c \neq 0 \\ \mathbf{R} & \text{if } c = 0 \end{cases}$$

(c) Finally, we ask, when is $f(x) = f^{-1}(x)$ for all $x$. If we set $f(x) = f^{-1}(x)$, we get

$$\frac{ax+b}{cx+d} = \frac{-dx+b}{cx-a}$$

If this is true for all $x$ in the domains, then

$$(ax + b)(cx - a) = (cx + d)(-dx + b)$$

which implies

$$c(a + d)x^2 - (a^2 - d^2)x - b(a + d) = 0.$$

The only way this can be satisfied for infinitely many real $x$ is if $c(a+d) = 0$, $a^2 - d^2 = 0$, and $b(a + d) = 0$. If $b \neq 0$ or $c \neq 0$, this implies $a = -d$ and the middle equation is also satisfied. If $b = c = 0$, then $a = \pm d$. Hence the $f$ for which $f = f^{-1}$ are all linear fractional mappings of the form

$$f(x) = \frac{ax + b}{cx - a}$$

with $b \neq 0$ or $c \neq 0$, and

$$f(x) = \pm x.$$