MATH 242 – Principles of Analysis
Selected Solutions for Problem Set 7, "B portion"
April 8, 2011

§3.5/15. A comment first: A fairly common idea that people tried to use in solutions for (a) was this. Let $0 < c < 1$. Then $0 < c^2 < 1$ as well. Since $f(0) < 0$ and $f(1) > 1$, this means c^2 is between $f(0)$ and $f(1)$. So by the Intermediate Value Theorem there is some $x_0 \in [0,1]$ where $f(x_0) = c^2$. The *problem with this* is that the question asked you to show that there is some c where $f(c) = c^2$. The x_0 above will not usually equal c. (Indeed that statement shows $\forall c \in [0,1], \exists x_0 \in [0,1]$ s.t. $f(x_0) = c^2$.) The correct way to apply the IVT here is to apply it *to a different function*. We want to find c where $f(c) = c^2$, so $f(c) - c^2 = 0$.

(a) Let $g(x) = f(x) - x^2$. By Theorem 3.4.10, this is continuous on $[0,1]$ since f is assumed to be continuous and x^2 is continuous everywhere. Using the assumptions on f, we have

$$g(0) = f(0) - 0 = f(0) < 0$$
$$g(1) = f(1) - 1 > 0$$

Hence since $k = 0$ is a value between $g(0)$ and $g(1)$, there is a $c \in (0,1)$ where $g(c) = f(c) - c^2 = 0$. This shows $f(c) = c^2$.

(b) Let $h(x) = f(x) - g(x)$. By Theorem 3.4.10, this is continuous on $[0,1]$ since f, g are assumed to be continuous. Since $f(0) < 0$, $g(0) \geq 0$, $f(1) > 1$, and $g(1) \leq 1$, we have

$$h(0) = f(0) - g(0) < 0$$
$$h(1) = f(1) - g(1) > 0$$

Hence since $k = 0$ is a value between $h(0)$ and $h(1)$, there is a $c \in (0,1)$ where $h(c) = f(c) - g(c) = 0$. This shows $f(c) = g(c)$.

22. This is FALSE. The function

$$f(x) = \frac{\cos(\pi/x)}{x}$$

on $(0,1]$ is one counterexample. Notice that at $x_n = \frac{1}{n}$ we have

$$f(1/n) = n\cos(n\pi) = (-1)^n n.$$

Since $x_n \to 0$ as $n \to \infty$, the interval $(0,1]$ contains points where f takes both positive and negative values that are arbitrarily large in absolute value:

$$f(1/n) = \begin{cases} n & \text{if } n \text{ is even} \\ -n & \text{if } n \text{ is odd} \end{cases}$$

It follows that f has no maximum or minimum on $(0, 1]$.

23. (a) This is FALSE. There are many non-continuous functions that have the "intermediate value property" (that is, for which the conclusion of the IVT holds). For example, on $[0, 1]$, the function

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2x - 1 & \text{if } 1/2 < x \le 1 \end{cases}$$

has a jump discontinuity at $x = 1/2$. But it takes every value between $f(0) = 0$ and $f(1) = 1$ at least once (most of them twice, in fact).

(b) This is FALSE. Here's a counterexample, and an explanation: Let

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and consider the interval $[a, b] = [-1, 1]$. First, f fails to be continuous at 0 because $\lim_{x\to 0} f(x)$ does not exist. But f *does have the "intermediate value property" on every subinterval* $[c_1, c_2] \subset [-1, 1]$. We can see this as follows. First, if $0 \ne [c_1, c_2]$, then f is actually continuous on $[c_1, c_2]$ and the IVT applies. Next, if $0 \in [c_1, c_2]$, then since f oscillates more and more quickly as $x \to 0$, f must take every value between -1 and 1 on that interval. This certainly implies that f takes every value between $f(c_1)$ and $f(c_2)$, since those are also in the range $[-1, 1]$.

Comment: A lot of people tried to construct a counterexample f by somehow using a pair c_1 and c_2 in the *definition* of f. That *cannot work* here because the property we want is that f is not continuous, but *for all* $a \le c_1 < c_2 \le b$, the intermediate value property holds for f on $[c_1, c_2]$. For instance if you tried to make a counterexample with jump discontinuities at particular c_1 and c_2, then on closed subintervals straddling one of those discontinuities there can be some y-values that are omitted by f. So the intermediate value property would fail for some intervals of x's.

§3.6/4 *Comment:* Many people had the correct idea on this one after talking to me in office hours. However, in order to give a complete proof that

$$f(x) = \sin(2\pi[x]x)$$

is not uniformly continuous on $\mathbf{R}$, you have to supply more details than many people did. Here is a complete argument.

Aiming for a contradiction, suppose f is uniformly continuous on $\mathbf{R}$. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon$ for all x, x' with $|x - x'| < \delta$. However, take $\varepsilon = 1$. Assuming there is a $\delta > 0$ that works here, we can find $n \in \mathbf{N}$ such that $\frac{1}{4n} < \delta$. Consider $x = n$ and $x' = n + \frac{1}{4n}$. Then

$$|x - x'| = \frac{1}{4n} < \delta,$$

but $f(n) = \sin(2\pi n^2) = 0$ and

$$\begin{aligned} f\left(n + \frac{1}{4n}\right) &= \sin\left(2\pi \left[n + \frac{1}{4n}\right]\left(n + \frac{1}{4n}\right)\right) \\ &= \sin\left(2\pi n\left(n + \frac{1}{4n}\right)\right) \\ &= \sin\left(2\pi n^2 + \frac{\pi}{2}\right) \\ &= 1 \end{aligned}$$

Hence $|f(x) - f(x')| = 1$, which is not less than $\varepsilon = 1$. This contradiction shows that f is not uniformly continuous on $\mathbf{R}$.

§4.1/11. *Comment:* Several people overlooked the fact that the statement to be proved is an "if and only if" statement, so both implications must be shown.

$\Rightarrow$: Suppose $h(x)$ is differentiable at c. Then the limit of the difference quotient of h:

$$h'(c) = \lim_{x \to c} \frac{h(x) - h(c)}{x - c}$$

must exist. This is equivalent to saying the two one-sided limits

$$\lim_{x \to c^-} \frac{h(x) - h(c)}{x - c}, \quad \lim_{x \to c^+} \frac{h(x) - h(c)}{x - c}$$

exist and *are equal*. By the definition of h, though,

$$\lim_{x \to c^-} \frac{h(x) - h(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c},$$

and that limit exists and equals $f'(c)$ since we assume f is differentiable at c. Similarly,

$$\lim_{x \to c^+} \frac{h(x) - h(c)}{x - c} = \lim_{x \to c^+} \frac{g(x) - g(c)}{x - c},$$

and that limit exists and equals $g'(c)$ since we assume g is differentiable at c. It follows that $f'(c) = g'(c)$.

$\Leftarrow$: Conversely, suppose $f'(c) = g'(c)$. Then taking the limits of the difference quotients on only one side, we can say:

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{g(x) - g(c)}{x - c}.$$

Because of the way h is defined piecewise, the first limit here is the same as

$$\lim_{x \to c^-} \frac{h(x) - h(c)}{x - c}$$

and the second is

$$\lim_{x \to c^+} \frac{h(x) - h(c)}{x - c}.$$

This shows that the two one-sided limits

$$\lim_{x \to c^-} \frac{h(x) - h(c)}{x - c}, \quad \lim_{x \to c^+} \frac{h(x) - h(c)}{x - c}$$

exist and *are equal*. Therefore,

$$h'(c) = \lim_{x \to c} \frac{h(x) - h(c)}{x - c}$$

must exist, so h is differentiable at c.