Mathematics 242 – Principles of Analysis
Selected Solutions – Problem Set 6, "B" Portion
March 25, 2011

A General Comment: The typical solution that many of you are submitting now for many of the proof-type problems is more like *a description of how you (or we :)) came to find the idea for the proof than what the proof itself should look like.* Some are too wordy, some do not explain enough, and some are not organized correctly. From this point on in the semester we will be working on moving beyond this level to practice the way mathematicians actually present proofs like this. That style might appear to be terse and uninformative at times, but it is the standard, so you need to learn how to do it.

I think one reason for the style that your proofs are "coming out in" is that some of you are attempting to do the scratch work for the problem set on the same set of sheets that you eventually hand in. Unless you are a real wiz at this sort of mathematics, that is not going to work for some problems – especially the more complicated proofs. What this means of course, is that you will probably want to start *delaying the final writeup on problem sets until you have worked out the ideas, and only write things up after you understand the ideas. You may* **want to keep both the notes you made "on the way" to the proof and the final product**. *That way you can reconstruct how you got to the final state if you need to.*

§3.2/3. To show: $\lim_{x\to c} x^n = c^n$ directly from the definition.

Proof: Given ε, let $\delta = \min\left(1, \frac{\varepsilon}{(n+1)(|c|+1)^{n-1}}\right)$. Then for all x with $0 < |x-c| < \delta$, we have $|x-c| < 1$, so $|x| < |c|+1$. From the factorization of a difference of nth powers and the triangle inequality,

$$\begin{aligned} |x^n - c^n| &= |x-c||x^{n-1} + x^{n-2}c + \cdots + c^{n-1}| \\ &\le |x-c|\left(|x|^{n-1} + |x|^{n-2}|c| + \cdots + |c|^{n-1}\right). \\ &\le |x-c|(n+1)(|c|+1)^{n-1} \end{aligned}$$

But then for all x with $|x-c| < \delta$, this is:

$$\begin{aligned} &< \frac{\varepsilon}{(n+1)(|c|+1)^{n+1}}(n+1)(|c|+1)^{n-1} \\ &= \varepsilon. \end{aligned}$$

This shows that $\lim_{x\to c} x^n = c^n$.

7. We assume that $f(x) \ge 0$ on some deleted neighborhood of c.

(a) To show: If $\lim_{x\to c} f(x) = 0$, then $\lim_{x\to c} \sqrt{f(x)} = 0$.

Proof: Since $\lim_{x\to c} f(x) = 0$, for all $\varepsilon > 0$, there exist corresponding $\delta > 0$ such that $|f(x) - 0| = f(x) < \varepsilon^2$ for all x with $0 < |x-c| < \delta$. But then for the same δ and all these x, $|\sqrt{f(x)} - 0| = \sqrt{f(x)} < \varepsilon$ (where we have used §1.3 Exercise 8). This shows $\lim_{x\to c} \sqrt{f(x)} = 0$.

Comment: Notice how this proof is laid out: It is a linear, step-by-step demonstration that the definition of $\lim_{x\to c} \sqrt{f(x)} = 0$ is satisfied. The first statement is OK since the definition of the limit says "for all positive numbers ... ." This is why we can get $f(x) < \varepsilon^2$ for any given $\varepsilon > 0$. This style is what you should be aiming for. I cut out all of the explanations (except the justifications for the individual steps) and just worked through to the conclusion $|\sqrt{f(x)} - 0| < \varepsilon$ for the suitable δ.

(b) Now we will show that if $\lim_{x\to c} f(x) = L > 0$, then $\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}$.

Proof: By the assumption on the limit of f, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x)-L| < \varepsilon\sqrt{L}$ for all x with $0 < |x-c| < \delta$. For that same δ and all x with $0 < |x-c| < \delta$, using the difference of squares factorization,

$$\begin{aligned}
|\sqrt{f(x)} - \sqrt{L}| &= \frac{|f(x) - L|}{|\sqrt{f(x)} + \sqrt{L}|} \\
&\leq \frac{|f(x) - L|}{|\sqrt{L}|} \quad (\text{since} f(x) \geq 0) \\
&\leq \frac{\varepsilon\sqrt{L}}{\sqrt{L}} \\
&= \varepsilon.
\end{aligned}$$

This shows $\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}$.

§3.4/3. (a) We assume f and g are continuous on an interval I containing c. If $f(c) \neq g(c)$, then we either have $f(c) > g(c)$ or vice versa. We will show that $h(x) = \max(f(x), g(x))$ and $k(x) = \min(f(x), g(x))$ are continuous at c under the assumption $f(c) > g(c)$. If the reverse inequality is true, the proof is similar but the maximum and minimum are interchanged. By problem 2, we know that if $f(c) > g(c)$, then there exists an interval I' contained in I and containing c such that $f(x) > g(x)$ for all $x \in I'$. This means that $f(x) = h(x)$ and $g(x) = k(x)$ for all $x \in I'$. Since the limits depend only on the values of the function near c, this means (using continuity of f and g):

$$\lim_{x\to c} h(x) = \lim_{x\to c} f(x) = f(c) = h(c)$$

and

$$\lim_{x\to c} k(x) = \lim_{x\to c} g(x) = g(c) = k(c).$$

Hence both h and k are continuous at c by definition.

Now suppose $f(c) = g(c) = b = h(c) = k(c)$. Since f and g are continuous at c, given $\varepsilon > 0$, there exist δ_1 and δ_2 such that $|f(x) - b| < \varepsilon$ for all x with $|x - c| < \delta_1$ and

$|g(x) - b| < \varepsilon$ for all x with $|x - c| < \delta_2$. If $\delta = \min(\delta_1, \delta_2)$, then for all x with $|x - c| < \delta$, we have both

$$|f(x) - b| < \varepsilon \quad \text{and} \quad |g(x) - b| < \varepsilon.$$

This implies

$$|h(x) - b| < \varepsilon \quad \text{and} \quad |k(x) - b| < \varepsilon,$$

since for every x, $h(x)$ is either $f(x)$ or $g(x)$ and similarly for $k(x)$. This shows that both h and k are continuous at c.

Comment: Although it is tempting to try to do something like what we did above in the other part of (a), to get a general proof in the case $f(c) = g(c)$, we *really do need* to proceed by the different argument used here. The reason is that *there need not be any intervals containing c on which the maximum or minimum is given by the same function between the two possibilities f or g. In particular, the graphs of f and g can cross at x's arbitrarily close to c so the max and min can be changing infinitely often in all intervals containing c.* For an example of this type, consider $f(x) = 0$ and $g(x) = x \sin(1/x)$ if $x \neq 0$, with $g(0)$ defined as $g(0) = 0$. Then the minimum and maximum alternate between f and g infinitely often in every interval containing $c = 0$.

(b) For this part, we will use the problem in Chapter 1 to give different formulas for the maximum and minimum:

$$h(x) = \max(f(x), g(x)) = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}$$

and

$$k(x) = \min(f(x), g(x)) = \frac{f(x) + g(x)}{2} - \frac{|f(x) - g(x)|}{2}.$$

We want to show this implies h and k are continuous at c, under the assumption that f and g are continuous at c. This follows from Theorems 3.4.10 and 3.4.11, since we are taking sums, differences, and compositions of continuous functions to get these formulas for the maximum and minimum.