Mathematics 242 – Principles of Analysis
Selected Solutions for Problem Set 5, "B" portion
March 18, 2011

§2.3/15. The idea of this problem is to show that if we assume that every bounded monotone sequence converges, then it follows that every nonempty set A that is bounded above has a least upper bound. (In other words, we will show that the statement of the Completeness Axiom follows from the Monotone Convergence Theorem. We proved the opposite implication in class, and both of these statements are also equivalent to the Nested Interval Theorem(!)) So we start with a set $A \subset \mathbf{R}$ that is bounded above. We construct two sequences x_n and y_n as described in the problem. We take $x_1 \in A$ an arbitrary element and we let y_1 in $\mathbf{R}$ be any upper bound for A. Then assuming x_n and y_n have already been constructed, let $z_n = \frac{x_n+y_n}{2}$ and consider two possible cases for determining x_{n+1} and y_{n+1}:

- Case 1: If z_n is an upper bound for A, then we let $x_{n+1} = x_n$ and $y_{n+1} = z_n$.
- Case 2: If, on the other hand z_n is not an upper bound for A, then there exists some $a' \in A$ with $a' > z_n$, so we let $x_{n+1} = a'$ and $y_{n+1} = y_n$.

We note the following:

- By construction, $x_n \in A$ for all n and y_n is an upper bound of A for all n.
- The sequence $\{x_n\}$ is monotone increasing and bounded above by y_1 since all $x_n \in A$.
- The sequence $\{y_n\}$ is monotone decreasing and bounded below by any x_m.

(a) It follows now that $\{x_n\}$ and $\{y_n\}$ are both convergent sequences, since our hypothesis is that every bounded monotone sequence in $\mathbf{R}$ converges.

(b) Since z_n is the midpoint of the line segment from x_n to y_n for each n, we note that for all $n \geq 1$,

$$y_{n+1} - x_{n+1} \leq \frac{1}{2}(y_n - x_n).$$

(We have equality in Case 1 and the strict inequality in Case 2 as above.) If we "unwind" this back to the beginning of the construction, we see

$$\begin{aligned} y_{n+1} - x_{n+1} &\leq \frac{1}{2}(y_n - x_n) \\ &\leq \frac{1}{2^2}(y_{n-1} - x_{n-1}) \\ &\vdots \\ &\leq \frac{1}{2^n}(y_1 - x_1). \end{aligned}$$

Hence if we let $n \to \infty$, we see that

$$\lim_{n\to\infty}(y_n - x_n) = 0,$$

since $\frac{1}{2^n} \to 0$ as $n \to \infty$. That implies

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$$

by part (a) and Theorem 2.2.5a.

(c) Call the limit of $\{x_n\}$ and $\{y_n\}$ $\alpha \in \mathbf{R}$. We want to show that α satisfies the properties of the least upper bound of A. First, since every y_n is an upper bound for A, we have $x \le y_n$ for all $x \in A$ and all n. It follows from Theorem 2.2.7 that $x \le \alpha$ for all $x \in A$. Hence α is an upper bound for A. On the other hand, if $\varepsilon > 0$, then since $x_n \to \alpha$, there exist $n_0 \in \mathbf{N}$ such that $\alpha - \varepsilon < x_n \le \alpha$ for all $n \ge n_0$. Since the $x_n \in A$ for all n, this implies that $\alpha - \varepsilon$ is not an upper bound for A. It follows that $\alpha = \mathrm{lub}(A)$.

§2.5/12. Both parts of this are TRUE. For part (a) suppose that we have an sequence of positive numbers $\{x_n\}$ with $x_n \to 0$ as $n \to \infty$. Then we want to show how to produce a strictly decreasing subsequence. We can take $x_{n_1} = x_1$ to start. Now suppose we have found

$$x_{n_1} > \cdots > x_{n_k}.$$

Letting $\varepsilon = x_{n_k}$ in the definition of convergence we see that there exists some N_0 such that $|x_n - 0| = x_n < \varepsilon = x_{n_k}$ for all $n \ge N_0$. Letting n_{k+1} be any one such integer that is also $> n_k$ we get $x_{n_{k+1}}$ satisfying

$$x_{n_1} > \cdots > x_{n_k} > x_{n_{k+1}}.$$

By induction, this shows there exists a strictly decreasing subsequence.

Now for (b), if infinitely many $x_n > 0$, then we can apply the same reasoning as in (a) to get a strictly decreasing subsequence chosen from the positive terms in the whole sequence. If there are not infinitely many $x_n > 0$, then there must be infinitely many $x_n < 0$ and we can repeat essentially the same argument above *with the negative terms.* We can let n_1 be any index such that $x_{n_1} < 0$. Then assuming we have found

$$x_{n_1} < \cdots < x_{n_k},$$

let $\varepsilon = |x_{n_k}|$ in the definition of convergence. we see that there exists some N_0 such that $|x_n - 0| = |x_n| < \varepsilon = |x_{n_k}|$ for all $n \ge N_0$. Letting n_{k+1} be any one such integer that is also $> n_k$ and such that $x_{n_{k+1}} < 0$, we get $x_{n_{k+1}}$ satisfying $|x_{n_{k+1}}| < |x_{n_k}|$, which implies $x_{n_{k+1}} > x_{n_k}$ (since both are negative). Hence

$$x_{n_1} < \cdots < x_{n_k} < x_{n_{k+1}}.$$

By induction, this shows there exists a strictly increasing subsequence.

13. We suppose that $\{x_n\}$ is a bounded sequence with the property that every convergent subsequence converges to the same limit a. Aiming for a contradiction, suppose $x_n \not\to a$.

This is equivalent to saying there exists some $\varepsilon_0 > 0$ such that for all $n_0 \in \mathbf{N}$, $|x_n - a| \geq \varepsilon_0$ for some $n \geq n_0$.

Let $\varepsilon_0 > 0$ be any fixed number for which it is true that

(*) *for all* $n_0 \in \mathbf{N}$, $|x_n - a| \geq \varepsilon_0$ *for some* $n \geq n_0$.

We claim that there is a subsequence $\{x_{n_k}\}$ with the property that $|x_{n_k} - a| \geq \varepsilon_0$ for all $k \in \mathbf{N}$. To see this, we apply (*) above with $n_0 = 1$, and we get a first term x_{n_1} for $n_1 \geq 1$ satisfying $|x_{n_1} - a| \geq \varepsilon_0$. Then assuming that $x_{n_1}, \ldots, x_{n_k}$ have been constructed, apply the statement (*) above with $n_0 = n_k + 1$. We get $n_{k+1} \geq n_k + 1$ such that $|x_{n_{k+1}} - a| \geq \varepsilon_0$. Continuing in the same way (i.e. by induction) this shows our claim.

It is clear that the subsequence $\{x_n\}$ constructed here cannot converge to a since all the terms are uniformly bounded away from a. Moreover, no subsequence of this subsequence can converge to a either, for the same reason.

However, since the original sequence was bounded, this subsequence is bounded too. Hence the Bolzano-Weierstrass theorem applies and shows that there must be a convergent subsequence of the $\{x_{n_k}\}$. That convergent subsequence is also a subsequence of the original $\{x_n\}$. Hence the assumption implies it must converge to a. This contradicts what we said in the last paragraph.

Hence x_n must converge to a. (Wow!)