$11 + 26 + 6 = 43$ total

PS4-A 2.1 / 1, 13, 15; 2.2 / 2, 3, 6ace, 7, 17ac; 2.3 / 2.

2.1 /

For each part, 1 for converges or not, 1 for reason

6 (1) only (a) converges to 0

In (b), there are n divisible by $500 >$ all $n_0 \in \mathbb{N}$, so never get $|y_n - 0| < \varepsilon$ for all $n \geq n_0$. Similarly for (c), the set of numbers $S = \{6, 66, 666, \ldots\}$ is not bounded, so there are $n \in S$ with $n \geq n_0$ all n_0.

1 for counterexample

13. 2 (a) Nothing: 1 $x_n = 7$ for all n converges

$$x_n = \begin{cases} 7 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$ does not

2 (b) If $|y_n - 3| \longrightarrow 4$, but $y_n \geq 0$ for all n, then $\{y_n\}$ converges to 7. Note that $|x - 3| = 4$ means $x \in \{7, -1\}$. However $|y_n - (-1)| < \varepsilon$ is impossible if $\varepsilon \leq 1$.

1 15. (b) is correct

2.2 /

2. (a) False: 1 Counterexample 1 $x_n = \begin{cases} n & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even} \end{cases}$

(b) True 1: $\{x_n\}$ is not bounded means for all $M > 0$, there exists n such that $|x_n| \geq M$. We can ignore $n \leq 10^6$ since all x_n with $n \leq 10^6$ satisfy $|x_n| \leq \max(|x_1|, \ldots, |x_n|)$. 1

or: (by contradiction) Suppose there exists M such that for all $n > 10^6$, $|x_n| \le M$. then $\{x_n\}$ is bounded since $|x_n| \le \max(M, |x_i|: i = 1, \dots, 10^6)$ for all n.

3. (a) True : 1 proof For all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_n| < \varepsilon$ for all $n \ge n_0$. this implies $|1/x_n| > 1/\varepsilon$. So $\{1/x_n\}$ is unbounded. 1

(b) False : 1 Counterexample 1 $x_n = \begin{cases} n & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd} \end{cases}$

6. (a) $\lim_{n \to \infty} \frac{1 + 3n - n^2}{4n^2 + \sqrt{n} + 1} = \lim_{n \to \infty} \frac{\frac{1}{n^2} + \frac{3}{n} - 1}{4 + \frac{1}{n^{3/2}} + \frac{1}{n^2}}$ 1

$= \boxed{-\frac{1}{4}}$ 1 by th. 2.25 (a) (c)

(c) $\lim_{n \to \infty} \frac{3^n - \pi^n}{4^n + 3^n} = \lim_{n \to \infty} \frac{(3/4)^n - (\pi/4)^n}{1 + (3/4)^n}$ 2 $= 0$

(e) $\lim_{n \to \infty} \frac{(2^n + 3^{n/2})^2}{4^n} = \lim_{n \to \infty} \frac{4^n + 2 \cdot 2^n \cdot 3^{n/2} + 3^n}{4^n}$

$= \lim_{n \to \infty} 1 + 2\left(\frac{2\sqrt{3}}{4}\right)^n + \left(\frac{3}{4}\right)^n$

2

$= \boxed{1}$

7. (a) $\lim_{n \to \infty} \frac{1}{n^5} \sum_{k=0}^{5} \binom{n}{k} 2^k$

we know $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!}$$

so $\binom{n}{0} = 1$

$$\binom{n}{1} = n$$

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$$

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24}$$

$$\binom{n}{5} = \frac{n(n-1)(n-2)(n-3)(n-4)}{120}$$

and $\displaystyle\lim_{n\to\infty} \frac{1}{n^5}\binom{n}{k} = \begin{cases} 0 & \text{if } 0 \le k \le 4 \\ \frac{1}{120} & \text{if } k = 5 \end{cases}$

Hence $\displaystyle\lim_{n\to\infty} \frac{1}{n^5}\sum_{k=0}^{5}\binom{n}{k}2^k = \frac{32}{120} = \frac{32}{8\cdot 15}$

$$= \boxed{\frac{4}{15}}.$$

4

c

(b) $$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} k$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{1}{2} + \frac{1}{2n}$$

$$= \boxed{\frac{1}{2}}$$ 4

17.(a) $$\lim_{n\to\infty} \sqrt{\frac{2n^2+1}{3n^2+1}} = \lim_{n\to\infty} \sqrt{\frac{2+\frac{1}{n^2}}{3+\frac{1}{n^2}}} = \sqrt{\frac{2}{3}}$$ 2

(c) $$\lim_{n\to\infty} \left(\sqrt{4n+3} - \sqrt{4n}\right) \cdot \frac{\left(\sqrt{4n+3} + \sqrt{4n}\right)}{\left(\sqrt{4n+3} + \sqrt{4n}\right)}$$

$$= \lim_{n\to\infty} \frac{3}{\sqrt{4n+3} + \sqrt{4n}} = 0.$$ 2

§ 2.3 /

1

2. (a) True: For all n, $x_{n+1} > x_n$ and $y_{n+1} > y_n$, so $x_{n+1} + y_{n+1} > x_n + y_n$.

1

1

(b) False: Counterexample: let $x_n = n^2$, $y_n = -\frac{1}{n}$. then (both increasing)

1 $x_n \cdot y_n = -n$ is decreasing.

Counterexample:

1 1

(c) False: Let $x_n = 4n$ (monotone inc.)

$y_n = -3 - n^2$ (monotone dec.)

(d)

but $x_n + y_n = -3 + 4n - n^2$ is not monotone

$n = 1 \quad x_1 + y_1 = 0$

$n = 2 \quad x_2 + y_2 = 1$

$n = 3 \quad x_3 + y_3 = 0.$