MATH 242 – Principles of Analysis
Selected Solutions – Problem Set 3 B
February 18, 2011

General Comment: The idea on problems 7, 8, 9 is to use the definition of the lub of a set of real numbers to prove the required assertion. So you need to show two things: (1) the given number is *an upper bound* for the set in question, and (2) no smaller number is also an upper bound (or equivalently that if you have any upper bound for the set, then it is greater than or equal to the "candidate" least upper bound.

§1.5/7 (a) Given that A is a nonempty bounded set of real numbers, and $B = \{x + k \mid x \in A\}$. Let $\alpha = \text{lub}(A)$.

(1) We claim that $\alpha + k$ is an upper bound for B. Since α is an upper bound for A, if $x \in A$ is any element of A, then $x \leq \alpha$. Hence, by Axiom 8 (p. 13), $x + k \leq \alpha + k$. This shows that $\alpha + k$ is an upper bound of B.
(2) Let β be any upper bound for B. Then $\beta \geq x + k$ for all $x \in A$ since the numbers $x + k$ for $x \in A$ are all the elements of B. Hence if we subtract k from both sides of the last ineqality (Axiom 8 again), $\beta - k \geq x$ for all $x \in A$. This says that $\beta - k$ is an upper bound for A. Hence, by the definition of the least upper bound, $\beta - k \geq \alpha$. Then, adding k to both sides in the last inequality (Axiom 8, yet again!), $\beta \geq \alpha + k$.

Therefore, by the definition $\alpha + k = \text{lub}(A) + k = \text{lub}(B)$.

Part (b) is almost the same, but least upper bounds are replaced by greatest lower bounds, and all inequalities reverse.

8. (a) First consider the case where $k = 0$. Then $B = \{0 \cdot x \mid x \in A\} = \{0\}$. Hence $\text{lub}(B) = 0 = 0 \times \text{lub}(A)$. Next, assume $k > 0$ and write $\alpha = \text{lub}(A)$ (for convenience).

(1) We claim that αk is an upper bound for B. Since α is an upper bound for A, if $x \in A$ is any element of A, then $x \leq \alpha$. Hence, since $k > 0$, by Axiom 9 (p. 13), $kx \leq k\alpha$. This shows that $k\alpha$ is an upper bound of B.
(2) Let β be any upper bound for B. Then $\beta \geq kx$ for all $x \in A$ since the numbers kx for $x \in A$ are all the elements of B. Since $k > 0$, $1/k$ exists in $\mathbf{R}$ and $1/k > 0$ too. We can multiply through by $1/k$ on both sides of the last ineqality (Axiom 9 again) to get $\beta/k \geq x$ for all $x \in A$. This says that β/k is an upper bound for A. Hence, by the definition of the least upper bound, $\beta/k \geq \alpha$. Then, multiplying k on both sides in the last inequality (Axiom 9, yet again!), $\beta \geq k\alpha$.

Therefore, by the definition $k\alpha = k \cdot \text{lub}(A) = \text{lub}(B)$.

Part (b) is related to part (a) in the same way that problem 7 part (b) is related to problem 7 part (a).

(c) If $k < 0$, then $k \cdot \text{lub}(A) = \text{glb}(B)$ and $k \cdot \text{glb}(A) = \text{lub}(B)$, since multiplying by a negative number reverses inequalities (Theorem 1.3.1 part (i)).

9. A general comment first – I did not get the feeling that *anyone* (even those who had completely correct solutions) really understood the subtlety of this problem. The thing that makes it tricky is the condition you are working with:

(*) For each element $x \in A$, there exists some $y \in B$ such that $y \geq x$.

Statements like this one, involving the *quantifiers* "for all" and "there exists" (or "for some") are going to be our *constant concern, our "bread and butter"* from now on in this course, and you need to get a good feeling for what "nested quantifiers" like the ones here actually mean.

(a) The given condition *does not imply* that that each element of B is greater than or equal to every element in A. Example: Let $A = \{0, 1\}$ and $B = \{0, 1, 2\}$. Then for each $x \in A$ there is some element $y \in B$ such that $y \geq x$. (For $x = 0$, take $y =$ any element in B. For $x = 1$, take $y = 1$ or $y = 2$.) On the other hand $0 \in B$ is not greater than or equal to $1 \in A$.

This is a perfectly good counterexample for part (a). It *does not* show all of the possibilities for sets A and B satisfying this condition and the possible subtleties, though. To get a better feeling for what the condition can say in more complicated examples, consider the following sets:

$$A = \left\{1 - \frac{1}{n} \mid n \in \mathbf{N}\right\}$$

and

$$B = \left\{1 - \frac{2n+1}{2n(n+1)} \mid n \in \mathbf{N}\right\}.$$

Let's write $a_n = 1 - 1/n$, and $b_n = 1 - (2n+1)/(2n(n+1))$. It is easy to check that $a_n < a_{n+1}$ for all $n \geq 1$ and $b_n < b_{n+1}$ for all $n \geq 1$. Some algebra shows that

$$b_n = \frac{1}{2}(a_n + a_{n+1})$$

for all $n \geq 1$. So b_n is the *midpoint of the interval* from a_n to a_{n+1}. As a result, the elements of A and B are completely "interleaved" like this:

$$a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < \cdots$$

You should be able to see that both of these sequences are tending to the limit 1 as $n \to \infty$. However, *all* of these are strict inequalities. In particular:

> Condition (*) holds, since for every $n \geq 1$, the elements $a_n \in A$ and $b_n \in B$ satisfy $b_n \geq a_n$.

On the other hand (unlike the first "simple" example, in which we had finite sets)

> No single element of B is an upper bound for A, since for each $n \geq 1$, $a_{n+1} > b_n$.

Moreover,

> Both A and B have least upper bound $= 1$, but that number is not contained in either of the sets.

This example shows that even in a "complicated" example, the inequality from part (b) of the problem holds. Here is a general proof of that part:

§1.5/ 9 (b) Let A and B be sets satisfying condition (*) above. If $x \in A$ is any element of A, then there is some $y \in B$ (possibly depending on x) such that $x \leq y$. That y satisfies $y \leq \text{lub}(B)$. Hence by Axiom 7 (p. 13), for all $x \in A$, $x \leq \text{lub}(B)$. Therefore $\text{lub}(B)$ is an upper bound for the set A. It follows from the definition that $\text{lub}(A) \leq \text{lub}(B)$.//

It is instructive to think how this works in the "complicated" example above(!)

13 (a) This is FALSE – a counterexample is $A = \{-2, 1\}$. $\alpha = \text{lub}(A) = 1$, but

$$\text{lub}(\{|x| \mid x \in A\}) = \text{lub}(\{1, 2\}) = 2 \neq |\alpha|.$$

(The intuition is that if $x < 0$, then $|x| = -x$ and multiplying an inequality on elements of A by -1 to get $|x| = -x$ in that case will *reverse the inequality* as in Theorem 1.3.1 (i).)

(b) This is TRUE – a proof goes like this: First if $x \in A$ and $y \in B$, then by Axiom 8 (twice):

$$x + y \leq \alpha + y \leq \alpha + \beta.$$

Then by Axiom 7, $x + y \leq \alpha + \beta$, and $\alpha + \beta$ is an upper bound of $\{x + y \mid x \in A, y \in B\}$. For the rest of the proof, instead of showing that any other upper bound of the set of sums $x + y$ is greater than or equal to $\alpha + \beta$, it will be somewhat more convenient to show one of our equivalent forms of the definition. We will show that no number $< \alpha + \beta$ is an upper bound for the set of sums $x + y$. So let $\varepsilon > 0$. Then there exist $x \in A$ satisfying $\alpha - \varepsilon/2 < x \leq \alpha$ and similarly, there exist $y \in B$ satisfying $\beta - \varepsilon/2 < y \leq \beta$. If we add corresponding terms here (using Axioms 8 and 7),

$$(\alpha + \beta) - \varepsilon < x + y \leq \alpha + \beta.$$

This shows that

$$\alpha + \beta = \text{lub}(\{x + y \mid x \in A, y \in B\}).$$

(c) and (d) are FALSE for reasons similar to part (a). (e) is TRUE by problem 8 from §1.3 (recall that B consists of positive numbers only).

23. This is similar to the proof that $\sqrt{2}$ is irrational that we did in class.

Suppose that $\sqrt{p} = \frac{m}{n}$ for some $m, n \in \mathbf{Z}$ with $n \neq 0$. By unique factorization in $\mathbf{Z}$, we may assume the fraction $\frac{m}{n}$ is in *lowest terms* – namely that m, n have *no common factors.*

The equation $\sqrt{p} = \frac{m}{n}$ implies that $n^2 p = m^2$. But that says p divides $m^2 = m \cdot m$. Because of the algebraic fact given in the problem, we conclude that p divides m. So we can write $m = pk$ for some integer k. Substituting in, this says $n^2 p = p^2 k^2$. But then we can cancel p's on each side to obtain $n^2 = pk^2$. Now the same argument shows that p divides $n^2 = n \cdot n$, so p divides n. Since p divides m and n, the fraction was not in lowest terms, and this is a contradiction. Hence $\sqrt{p}$ is not rational.