MATH 242 – Principles of Analysis
Selected Solutions for Problem Set 2 – B portion
February 11, 2011

§1.3/

8. We must show that if $a, b > 0$, then $a < b$ if and only if $a^2 < b^2$. Since this is an if and only if statement, we must show the two implications. So first, assume $a < b$. Since $a > 0$, Axiom 9 for the real numbers implies $a^2 = a \cdot a < a \cdot b$. Similarly, since $b > 0$, $a \cdot b < b \cdot b = b^2$. But then by Axiom 7 (the transitive property), $a^2 < a \cdot b < b^2$ implies $a^2 < b^2$.

For the other implication, we start by assuming $a, b > 0$ and $a^2 < b^2$. Arguing by contradiction, if $a \geq b$, then by the same reasoning as in the other part (using Axioms 7 and 9), we would have $a^2 \geq b^2$. But that contradicts the hypothesis $a^2 < b^2$. So it must be true that $a < b$. (This is (essentially) the proof of part (e) of Theorem 1.3.4.)

9. (Comment: The problems that many people had on this one were issues of how to structure the proof; most people did have the right idea.)

Approach 1: We start from the observation that since $0 < a < b$, $2\sqrt{a}\sqrt{b-a} > 0$, and hence,

$$b < b + 2\sqrt{a}\sqrt{b-a}. \tag{1}$$

Adding and subtracting a on the right side, this implies that

$$(\sqrt{b})^2 = b < (b-a) + 2\sqrt{a}\sqrt{b-a} + a = (\sqrt{b-a} + \sqrt{a})^2.$$

But then by problem 8,

$$\sqrt{b} < \sqrt{b-a} + \sqrt{a},$$

and finally by Axiom 8, subtracting $\sqrt{a}$ from both sides,

$$\sqrt{b} - \sqrt{a} < \sqrt{b-a}.$$

(A question you might be asking: *how would any one know to start from the observation (1)?* The answer is that you might see this by doing the algebraic operations in the reverse of the order above. But the *proof* should start with the hypothesis and end with the statement to be shown. You cannot just state the thing to be proved and work from there, because that means you are *asserting it is true*(!))

Approach 2: To show that

$$\sqrt{b} - \sqrt{a} < \sqrt{b-a}, \tag{2}$$

it suffices by problem 8 to show that

$$(\sqrt{b} - \sqrt{a})^2 < (\sqrt{b-a})^2,$$

or equivalently that

$$b - 2\sqrt{a}\sqrt{b} + a < b - a. \tag{3}$$

(Note that we are not asserting that (2) or (3) *is true* here. We are reasoning about what it takes to show that (2) is true!) Now problem 8 also implies that if $a < b$, then $\sqrt{a} < \sqrt{b}$. Hence $2a = 2\sqrt{a}\sqrt{a} < 2\sqrt{a}\sqrt{b}$ (Axiom 9). But then $-2\sqrt{a}\sqrt{b} < -2a$ by Theorem 1.3.1 (i). Hence

$$b - 2\sqrt{a}\sqrt{b} + a < b - 2a + a = b - a.$$

This shows that (3) is true, and hence (2) is true as well.

§1.4/3. To show: For all $n \geq 0$ and all $x > -1$, $(1+x)^n \geq 1 + nx$.

Proof: (by induction on n). The base case here is given as $n = 0$. Since $x > -1$, we know $1 + x > 0$. Hence have $(1+x)^0 = 1 \geq 1 + 0x$, so the inequality is true for $n = 0$.

Now the induction hypothesis is

$$(1+x)^k \geq 1 + kx, \tag{4}$$

Consider $(1+x)^{k+1}$. Since $1 + x > 0$, from (4) and Axiom 9 we have

$$(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x).$$

Expanding out the term on the right shows

$$(1+x)^{k+1} \geq 1 + (k+1)x + kx^2.$$

Since k is one of the values of $n \geq 0$ and $x \in \mathbf{R}$, $kx^2 \geq 0$. Therefore

$$(1+x)^{k+1} \geq 1 + (k+1)x + kx^2 \geq 1 + (k+1)x.$$

The inequality we want follows from Axiom 7.

(Comment: The hypothesis $x > -1$ is used in the base case and the induction step. 0^0 is undefined, so $x = -1$ is not allowed in the base case. In the induction step, $1 + x > 0$ is used when we invoke Axiom 9.)