Problem Set 10: Solutions for Selected Problems

§6.1/1

(c) For this series, the partial sums satisfy

$S_{4k} = 0$ for all $k \geq 1$

$S_{4k+2} = 2$ for all $k \geq 0$.

Therefore $\lim_{n\to\infty} S_n$ does not exist, and the series diverges.

(d) Note that [ ] denotes the greatest integer function as always. Hence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\left[\left(\frac{n+1}{2}\right)\right]} = 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots$$

Hence the partial sums are $S_{2k} = 0$ all k and $S_{2k+1} = \frac{1}{k+1}$. This shows $\lim_{n\to\infty} S_n = 0$, so the series converges to 0.

(f) $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$ is a telescoping series.

We have $S_k = \left(1 - \cancel{\frac{1}{\sqrt{2}}}\right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}\right) + \cdots + \left(\cancel{\frac{1}{\sqrt{k}}} - \frac{1}{\sqrt{k+1}}\right)$

$$= 1 - \frac{1}{\sqrt{k+1}}$$

Hence $\lim_{k\to\infty} S_k = 1$, so the series converges.

4. (c) Geometric series with $a = 25$, $r = \frac{4}{9}$,

Sum is $\dfrac{25}{1-4/9} = \boxed{45}$.

(d) Geometric series with ratio $0 < e^{-2} < 1$, first term $= 1$.
Sum $= \dfrac{1}{1-e^{-2}} = \boxed{\dfrac{e^2}{e^2-1}}$

(e) Geometric series with ratio $\frac{\pi}{e} > 1$ $\therefore$ divergent

(g) Geometric series with ratio $-\frac{3}{10}$, first term $a = 20 \cdot \left(\frac{-3}{10}\right)^2 = \frac{9}{5}$. Since $\left|-\frac{3}{10}\right| < 1$, the series converges to $\dfrac{9/5}{1+\frac{3}{10}} = \boxed{\dfrac{18}{13}}$.

8. Consider the partial sums of the series $a_1 + b_1 + a_2 + b_2 + \cdots$. If $n = 2k$ is even, we have
$S_n = a_1 + b_1 + \cdots + a_k + b_k = (a_1 + a_2 + \cdots + a_k) + (b_1 + \cdots + b_k)$,
using associativity and commutativity of finite sums.
On the other hand, if $n = 2k+1$ is odd, then
$S_n = a_1 + b_1 + \cdots + a_k + b_k + a_{k+1} = (a_1 + \cdots + a_{k+1}) + (b_1 + \cdots + b_k)$
Either way, since we assume $\sum a_n = \alpha$, $\sum b_n = \beta$

$$\lim_{n\to\infty} S_n = \lim_{k\to\infty} \begin{cases} (a_1 + \cdots + a_k) \\ (a_1 + \cdots + a_{k+1}) \end{cases} + \lim_{k\to\infty} (b_1 + \cdots + b_k)$$

$$= \alpha + \beta.$$

(the "point" of this problem is that associativity and commutativity only hold for finite sums.)
necessarily

11. Using the integral test, we see $\frac{1}{x(\ln x)^p}$ is monotonically decreasing to 0 for $x \geq 2$. By u-substitution ($u = \ln x$)

$$\int (\ln x)^{-p} \frac{1}{x}\, dx = \int u^{-p}\, du$$

$$= \begin{cases} \dfrac{(\ln x)^{-p+1}}{-p+1} + C & \text{if } p \neq 1 \\ \ln|\ln x| + C & \text{if } p = 1. \end{cases}$$

Now $\int_2^B (\ln x)^{-p} \frac{1}{x}\, dx = \begin{cases} \dfrac{(\ln B)^{-p+1}}{-p+1} - \dfrac{(\ln 2)^{-p+1}}{-p+1} & \text{if } p \neq 1 \\ \ln(\ln B) - \ln(\ln 2) & \text{if } p = 1 \end{cases}$

the limit as $B \to \infty$ is finite only if $-p+1 < 0$ so $\boxed{p > 1}$

12. (a) By the integral test again, we need to consider $\lim_{B \to \infty} \int_1^B \arctan\left(\frac{1}{x}\right) dx$.

Using "parts" with $u = \arctan\left(\frac{1}{x}\right)$ $dv = dx$,

$$\int \arctan\left(\frac{1}{x}\right) dx = x \arctan\left(\frac{1}{x}\right) + \int \frac{x}{x^2+1}\, dx$$

$$= x \arctan\left(\frac{1}{x}\right) + \frac{1}{2} \ln|x^2+1| + C$$

and $\int_1^B \arctan\left(\frac{1}{x}\right) dx = B \arctan\left(\frac{1}{B}\right) + \frac{1}{2}\ln|B^2+1| - \frac{\pi}{4} - \frac{1}{2}\ln 2$

By L'Hospital's Rule, $\lim_{B\to\infty} B\arctan\left(\frac{1}{B}\right)$

$$= \lim_{B\to\infty} \frac{\arctan\left(\frac{1}{B}\right)}{\frac{1}{B}} = \lim_{B\to\infty} \frac{\frac{1}{1+\frac{1}{B^2}}\cdot \cancel{-\frac{1}{B^2}}}{\cancel{-\frac{1}{B^2}}} = 1.$$

But $\lim_{B\to\infty} \frac{1}{2}\ln|B^2+1| = +\infty$. therefore

$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n}\right)$ diverges.

§6.2 /

1 (a) – (g) all of these are similar (all conditionally convergent. For instance:

(g) Since $(-1)^n - (-1)^{n+1} = \begin{cases} 2 & \text{if } n \text{ even} \\ -2 & \text{if } n \text{ odd} \end{cases} = (-1)^n\cdot 2$

the series is $\sum_{n=1}^{\infty} \frac{(-1)^n\cdot 2}{n+1}$. Since $\frac{1}{n+1}$

decreases to 0 as $n\to\infty$, the alternating series test implies $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 2}{n+1}$ converges.

But $\frac{2}{n+1} \geq \frac{2}{2n} = \frac{1}{n}$ for all $n \geq 1$ and

$\sum \frac{1}{n}$ diverges. So $\sum_{n=1}^{\infty}\frac{2}{n+1}$ diverges

by the comparison test.

(i) $\left|\frac{(-1)^n \sin(n)}{n^{3/2}}\right| \leq \frac{1}{n^{3/2}}$ since $|\sin(n)| \leq 1$ all n.

By the comparison test, since $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges,

$\sum_{n=1}^{\infty} \frac{(-1)^n \sin(n)}{n^{3/2}}$ converges absolutely.

5.(d) $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(-1)^n (n!)^2} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)}$

$= \frac{1}{4}.$

therefore, the series converges absolutely by the Ratio Test.

8.(a) The Ratio Test does not apply to this series because $\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \frac{1}{2} & \text{if } n \text{ is even.} \end{cases}$

therefore $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist.

9 (a) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r < 1$, then for all $\varepsilon > 0$, there exist n_0 such that $\left| \sqrt[n]{|a_n|} - r \right| < \varepsilon$ for all $n \geq n_0$. If $\varepsilon < 1 - r$, then $\sqrt[n]{|a_n|} < r + \varepsilon < 1$ for all $n \geq n_0$. Let $R = r + \varepsilon$, so $R < 1$. Then $\sum_{n=n_0}^{\infty} R^n$ is a convergent geometric series and $|a_n| < R^n$ for all $n \geq n_0$. It follows from the Comparison test that $\sum_{n=n_0}^{\infty} |a_n|$ converges. It follows that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(c) For all p, $\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n\to\infty} \left(\frac{1}{n^{1/n}}\right)^p = 1.$

But $\sum \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$.

§6.3 / 2(c)

Using the Ratio Test, we consider

$$\lim_{n\to\infty} \left| \frac{(n+1)(x-3)^{n+1}}{(n+2)\, 3^{n+1}} \cdot \frac{(n+1)\, 3^n}{n\,(x-3)^n} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{n(n+2)} \frac{|x-3|}{3}$$

Since $\displaystyle\lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{2}{n}} = 1,$$

the series converges absolutely for all x with

$0 \le \frac{|x-3|}{3} < 1$, or $|x-3| < 3$, so $x \in (0,6)$.

At the end points of the interval:

$x=0$ gives $\displaystyle\sum_{n=0}^{\infty} \frac{n\,(-3)^n}{(n+1)\,3^n} = \sum_{n=0}^{\infty} (-1)^n \frac{n}{n+1}$.

this diverges by the n^{th} term test $\left(\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0\right)$

$x=6$ gives $\displaystyle\sum_{n=0}^{\infty} \frac{n\; 3^n}{(n+1)\, 3^n}$, which also diverges.

Therefore the set of convergence is $(0,6) =$
$\{x \in \mathbb{R} \mid 0 < x < 6\}$.